Calculation and Implementation of the Gradient of the DSO Objective Function for the General Acoustic Model

Alain Sei

CRPC-TR93303
March 1993

Center for Research on Parallel Computation
Rice University
P.O. Box 1892
Houston, TX 77251-1892
Calculation and Implementation of the Gradient
of the DSO Objective Function for the
General Acoustic Model

Alain Sei

March, 1993

TR93-05
Calculation and Implementation of the Gradient of the DSO Objective Function for the General Acoustic Model

ALAIN SEI

The Rice Inversion Project
Department of Computational and Applied Mathematics
Rice University
Houston, Texas 77005

March 1, 1993

Abstract
We present in this paper the computation of the DSO objective function in the general acoustic case. In this model, the density and the velocity are functions of the space variables. We use a pertubationnal approach, justified by the separation of scales between the long and short wavelength components of the model. An extension of the adjoint state technique yields an accurate expression of the gradient of the DSO objective function. Then we use a finite difference approximation of the wave equation, and give in the discrete case the expression of the gradient. We show in that case how to apply the principle of images, so that the discrete operators involved are self-adjoint and give exact discrete integration by parts.
1 The Forward Map

We consider the following wave propagation problem. Given an acoustic medium defined by its the density \( \rho(x, z) \) and its velocity \( c(x, z) \) for \((x, z) \in \Omega = [0; X] \times [0; Z]; t \in ]0; T[\), find the pressure field \( u(x, z, t) \) which is the solution of the radiation problem:

\[
\begin{cases}
\frac{1}{\rho c^2} \frac{\partial^2 u}{\partial t^2}(x, z, t) - \nabla \left( \frac{1}{\rho(x, z)} \nabla u(x, z, t) \right) = f(x, z, t; x_s, z_s) \\
u(x, z, 0) = \frac{\partial u}{\partial t}(x, z, 0) = 0 \\
u(0, 0, t) = u(X, z, t) = u(x, 0, t) = u(x, Z, t) = 0
\end{cases}
\]

(1)

where \((x_s, z_s)\) is the location of the source. To find solutions of this problem, we adopt a perturbationnal approach. We look for solutions 'close' to a reference solution \( u_0 \) given for a reference distribution of density \( \rho_0 \) and velocity \( c_0 \). We therefore suppose \( \rho \) and \( c \) to be given by

\[
\begin{cases}
\rho(x, z) = \rho_0(x, z) + \delta \rho(x, z) \\
c(x, z) = c_0(x, z) + \delta c(x, z)
\end{cases}
\]

(2)

We use in those expressions the separation of scales inherent to the physics of the problem. \( c_0(x, z) \) and \( \rho_0(x, z) \) are supposed to be smooth functions, and \( \delta c(x, z) \) and \( \delta \rho(x, z) \) are supposed to be oscillatory functions. We suppose that the perturbations are relatively small compared to the references that is:

\[
\| \frac{\delta \rho}{\rho_0} \| << 1 \quad \| \frac{\delta c}{c_0} \| << 1
\]

where \( \| . \| \) is a certain norm on the functional space where \( \rho \) and \( c \) are defined (e.g the \( L^2 \) norm). We will see as the calculation proceeds the necessary regularity of the different so called models.

Then we look for a solution written as \( u = u_0 + \delta u \) where \( u_0 \) verifies (1) and \( \delta u \) is the solution of the linearised problem:

\[
\begin{cases}
\frac{1}{\rho_0 c_0^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho_0} \nabla \delta u \right) = \frac{1}{\rho_0 c_0^2} \left( \frac{\delta \rho}{\rho_0} + 2 \frac{\delta c}{c_0} \right) \frac{\partial^2 u_0}{\partial t^2} - \nabla \left( \frac{\delta \rho}{\rho_0^2} \nabla u_0 \right) \\
\delta u(x, z, 0) = \frac{\partial \delta u}{\partial t}(x, z, 0) = 0 \\
\delta u(0, 0, t) = \delta u(X, z, t) = \delta u(x, 0, t) = \delta u(x, Z, t) = 0
\end{cases}
\]

(3)
We define the reflectivities (cf [1]), as the relative perturbations, that is:

\[ r_\rho = \frac{\delta \rho}{\rho_0} \quad r_c = \frac{\delta c}{c_0} \]

Supposing that \( f, r_\rho \) and \( r_c \) do not have overlapping supports, we can write (3) as follows

\[
\begin{aligned}
\frac{1}{\rho_0 c_0^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho_0} \nabla \delta u \right) &= (r_\rho + 2r_c) \nabla \left( \frac{1}{\rho_0} \nabla u_0 \right) - \nabla \left( \frac{r_c}{\rho_0} \nabla u_0 \right) \\
\delta u(x, z, 0) &= \frac{\partial \delta u}{\partial t}(x, z, 0) = 0 \\
\delta u(0, z, t) &= \delta u(X, z, t) = \delta u(x, 0, t) = \delta u(x, Z, t) = 0
\end{aligned}
\]

(4)

We are now able to define the forward map \( F \) of our inverse problem. It maps the functions defining the medium, the density \( \rho \), the velocity \( c \), the reflectivity in density \( r_\rho \) and the reflectivity in velocity \( r_c \) to the seismogram produced in this medium at the array of receivers \( (x_r, z_r)_{r=1,R} \) by the source \( f(t) \) located in \( (x_s, z_s)_{s=1,S} \).

Therefore we define the forward map by:

\[
F(\rho, c, r_\rho, r_c)(t; x_s, z_s) = \sum_{s=1}^{S} \sum_{r=1}^{R} u(x_r, z_r, t; x_s, z_s)
\]

where \( u \) satisfies:

\[
\begin{aligned}
\frac{1}{\rho c^2} \frac{\partial^2 u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u_0 \right) &= f(x, z, t; x_s, z_s) \\
\frac{1}{\rho c^2} \frac{\partial^2 u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u \right) &= (r_\rho + 2r_c) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{r_c}{\rho} \nabla u_0 \right) \\
\frac{\partial u}{\partial t}(x, z, 0) &= \frac{\partial u_0}{\partial t}(x, z, 0) = \frac{\partial u_0}{\partial t}(x, z, 0) = 0 \\
u_0(x, z, 0) &= u_0(x, z, 0) = \frac{\partial u_0}{\partial t}(x, z, 0) = \frac{\partial u_0}{\partial t}(x, z, 0) = 0 \\
u_0(0, z, t) &= u_0(x, 0, t) = u_0(X, z, t) = u_0(x, Z, t) = 0 \\
u(0, z, t) &= u(x, 0, t) = u(X, z, t) = u(x, Z, t) = 0
\end{aligned}
\]

(5)
2 The Objective Function and the Normal Equations

Following [2], we introduce the Differential Semblance Objective function as:

$$\tilde{J}(\rho, c) = \min_{\rho, c} J_{DS}(\rho, c, r_p, r_c)$$

$$J_{DS}(\rho, c, r_p, r_c) = \frac{1}{2} \left\{ ||F(\rho, c, r_p, r_c) - F_{data}||^2_{L^2(0,T)} + \sigma_p \left( \frac{\partial r_p}{\partial x_s} \right)^2 \right\}$$

$$+ \sigma_c \left( \frac{\partial r_c}{\partial x_s} \right)^2 \right\}$$

where $W$ is a regularizing operator, for instance $W = I$ or $W = \nabla_{x,s}$ (cf [3]). The first term of $J_{DS}$ fits the data and the other part of $J_{DS}$ enforces coherency in the inverted models (cf [3]). When we use $W = \nabla_{x,s}$ we need $r_p$ and $r_c$ to belong to $H^1(\Omega)$.

We see by the expression of the cost function alone, that we have two minimization problems to solve.

First the minimization on the so called 'inner variables' $r_p$ and $r_c$. Then once $r_p$ and $r_c$ determined at $\rho$ and $c$ fixed, we want to minimize $J$ over the so called 'outer variables' $\rho$ and $c$.

We start with the inner variables minimization. We want to compute the gradient of $J_{DS}$ with respect to $r_p$ and $r_c$. The first variation $\delta J_{DS}$ of $J_{DS}$ with respect to a variation $\delta r_p$ and $\delta r_c$ in $r_p$ and $r_c$ is given by:

$$\delta J_{DS} = (D_{r_p}F \cdot \delta \rho + D_{r_c}F \cdot \delta c, F - F_{data})_{L^2(0,T)} + \sigma_p^2 \left( \frac{\partial r_p}{\partial x_s}, \frac{\partial \delta r_p}{\partial x_s} \right)_{L^2(\Omega)}$$

$$+ \sigma_c^2 \left( \frac{\partial r_c}{\partial x_s}, \frac{\partial \delta r_c}{\partial x_s} \right)_{L^2(\Omega)} + \lambda_p^2 (W \delta r_p, W r_p)_{L^2(\Omega)} + \lambda_c^2 (W \delta r_c, W r_c)_{L^2(\Omega)}$$

where $D_{r_p}F$ is the derivative of $J_{DS}$ with respect to $r_p$ and $D_{r_c}F$ is the derivative of $J_{DS}$ with respect to $r_c$. This can be written as:

$$\delta J_{DS} = \left( \delta \rho, D_{r_p}F^*(F - F_{data}) - \sigma_p^2 \frac{\partial^2 r_p}{\partial x_s^2} + \lambda_p^2 W^T W r_p \right)_{L^2(\Omega)}$$

$$+ \left( \delta c, D_{r_c}F^*(F - F_{data}) - \sigma_c^2 \frac{\partial^2 r_c}{\partial x_s^2} + \lambda_c^2 W^T W r_c \right)_{L^2(\Omega)}$$

where $D_{r_p}F^*$ is the adjoint operator of the derivative $D_{r_p}F$ and $D_{r_c}F^*$ is the adjoint of
Calculation of the Gradient for the DSO Objective Function

the derivative $D_{r_p} F$. This simply means that:

\[
\begin{align*}
\nabla_{r_p} J_{DS} &= D_{r_p} F^*(F - F_{\text{data}}) - \sigma_p^2 \frac{\partial^2 r_p}{\partial x^2_p} + \lambda_p^2 W^T W r_p \\
\nabla_{r_c} J_{DS} &= D_{r_c} F^*(F - F_{\text{data}}) - \sigma_c^2 \frac{\partial^2 r_c}{\partial x^2_c} + \lambda_c^2 W^T W r_c
\end{align*}
\]

(7)

When we use $W = \nabla_{x,z}$, we have to suppose that $r_p$ and $r_c$ belong to $H^2(\Omega)$. Setting the gradients to zero in (7) we get the following normal equations:

\[
\begin{align*}
M_{r_p} F - \sigma_p^2 \frac{\partial^2 r_p}{\partial x^2_p} + \lambda_p^2 W^T W r_p &= M_{r_p} F_{\text{data}} \\
M_{r_c} F - \sigma_c^2 \frac{\partial^2 r_c}{\partial x^2_c} + \lambda_c^2 W^T W r_c &= M_{r_c} F_{\text{data}}
\end{align*}
\]

(8)

Now we need to know the effect of the operator $D_{r_p} F^* = M_{r_p}$ and $D_{r_c} F^* = M_{r_c}$ on some seismogram $\varphi(x_r, z_r, t; x_s, z_s)$. Since $F$ is linear in $r_p$ and $r_c$, we have:

\[
\begin{align*}
D_{r_p} F, \delta r_p &= F(\rho, c, \delta r_p, 0) \\
D_{r_c} F, \delta r_c &= F(\rho, c, 0, \delta r_c)
\end{align*}
\]

therefore

\[
(M_{r_p} \varphi, \delta r_p)_{L^2(\Omega)} = \sum_{s=1}^{S} \sum_{t=1}^{R} (\varphi(x_r, z_r; x_s, z_s), D_{r_p} F(\rho, c, r_p, r_c), \delta r_p)_{L^2(0,T)}
\]

\[
= \sum_{s=1}^{S} \sum_{t=1}^{R} (\varphi(x_r, z_r; x_s, z_s), F(\rho, c, r_p 0), \delta r_p)_{L^2(0,T)}
\]
We have $F(\rho, c, r_p, 0) = \delta u$ solution of:

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u \right) &= \delta r_p \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{\delta r_p}{\rho} \nabla u_0 \right) \\
\frac{1}{\rho c^2} \frac{\partial^2 u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u_0 \right) &= f(x, z, t; x_2, z_3) \\
\delta u(x, z, 0) &= \frac{\partial \delta u}{\partial t}(x, z, 0) = 0 \\
\delta u(0, z, t) &= \delta u(X, z, t) = \delta u(x, 0, t) = \delta u(x, Z, t) = 0 \\
u_0(x, z, 0) &= \frac{\partial u_0}{\partial t}(x, z, 0) = 0 \\
u_0(0, z, t) &= u_0(X, z, t) = u_0(x, 0, t) = u_0(x, Z, t) = 0
\end{align*}
\]

(9)

Following a well known technique (cf [5], [6], [7]), we define the adjoint $u_0$ state as follows:

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u_0 \right) &= \sum_{r=1}^{R} \varphi(x_r, z_r, t; x_2, z_3) \delta(x - x_2, z - z_3) \\
u_0(x, z, T) &= \frac{\partial u_0}{\partial t}(x, z, T) = 0 \\
u_0(0, z, t) &= u_0(X, z, t) = u_0(x, 0, t) = u_0(x, Z, t) = 0
\end{align*}
\]

(10)

then we have:

\[
(M_{r_p, \varphi}, \delta r_p)_{L^2(\Omega)} = \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \sum_{r=1}^{R} \varphi(x_r, z_r, t; x_2, z_3) \delta u(x, z, t; x_2, z_3) \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \sum_{r=1}^{R} \varphi(x_r, z_r, t; x_2, z_3) \delta u(x, z, t; x_2, z_3) \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \frac{1}{\rho c^2} \frac{\partial^2 u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u_0 \right) \right) \delta u(x, z, t; x_2, z_3) \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \frac{1}{\rho c^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u \right) \right) w_0(x, z, t; x_2, z_3) \, dx \, dz \, dt
\]
Calculation of the Gradient for the DSO Objective Function

\[ \begin{align*}
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \delta \tau_{r} \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) - \nabla \left( \frac{\delta \tau_{r}}{\rho} \nabla u_{0} \right) \right) w_{0}(x, z, t; x_s, z_s) \, dx \, dz \, dt
\end{align*} \]

\[ = \int_{\Omega} \int_{0}^{T} \sum_{s=1}^{S} \left( w_{0} \cdot \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) + \frac{1}{\rho} \nabla w_{0} \nabla u_{0} \right) dt \delta \tau_{r} \, dx \, dz \]

whence

\[ M_{r, \varphi} = \sum_{s=1}^{S} \int_{0}^{T} w_{0} \cdot \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) + \frac{1}{\rho} \nabla w_{0} \nabla u_{0} dt \]

in the same way we have:

\[ (M_{r, \varphi}, \delta r_{c})_{L^2(\Omega)} = \sum_{s=1}^{S} \sum_{r=1}^{R} \left( \varphi(x_{r}, z_{r}; x_s, z_s), D_{r, F}(\rho, c, r_{r}, r_{c}, \delta r_{c})_{L^2(0; T)} \right) \]

\[ = \sum_{s=1}^{S} \sum_{r=1}^{R} \left( \varphi(x_{r}, z_{r}; x_s, z_s), F(\rho, c, 0, r_{c}, \delta r_{c})_{L^2(0; T)} \right) \]

We have \( F(\rho, c, 0, r_{c}) = \delta u \) solution of:

\[ \begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u \right) &= 2 \delta \tau_{c} \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) \\
\frac{1}{\rho c^2} \frac{\partial^2 u_{0}}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) &= f(x, z, t; x_s, z_s) \\
\delta u(x, z, 0) &= \frac{\partial \delta u}{\partial t}(x, z, 0) = 0 \\
\delta u(0, z, t) &= \delta u(X, z, t) = \delta u(x, 0, t) = \delta u(x, Z, t) = 0 \\
u_{0}(x, z, 0) &= \frac{\partial u_{0}}{\partial t}(x, z, 0) = 0 \\
u_{0}(0, z, t) &= u_{0}(X, z, t) = u_{0}(x, 0, t) = u_{0}(x, Z, t) = 0
\end{align*} \]

(11)

then we have:

\[ (M_{r, \varphi}, \delta r_{c})_{L^2(\Omega)} = \sum_{s=1}^{S} \int_{0}^{T} \sum_{r=1}^{R} \varphi(x_{r}, z_{r}; x_s, z_s) \delta u(x, z, t; x_s, z_s) dt \]
\[ \begin{align*}
\sum_{s=1}^{S} \int_{0}^{T} \sum_{r=1}^{R} \phi(x, r, t; x_{s}, z_{s}) \delta(x - x_{s}, z - z_{s}) \delta u(x, z, t; x_{s}, z_{s}) \,dx \,dz \,dt &= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \frac{1}{\rho c^2} \frac{\partial^2 w_{0}}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla w_{0} \right) \right) \delta u(x, z, t; x_{s}, z_{s}) \,dx \,dz \,dt \\
&= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \frac{1}{\rho c^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u \right) \right) w_{0}(x, z, t; x_{s}, z_{s}) \,dx \,dz \,dt \\
&= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( 2 \delta r_{c} \nabla \left( \frac{1}{\rho} \nabla w_{0} \right) \right) w_{0}(x, z, t; x_{s}, z_{s}) \,dx \,dz \,dt \\
&= \int_{\Omega} \left( \int_{0}^{T} \sum_{s=1}^{S} 2 w_{0} \nabla \left( \frac{1}{\rho} \nabla w_{0} \right) dt \right) \delta r_{c} \,dx \,dz
\end{align*} \]
whence
\[ \begin{align*}
M_{r_{c}} \phi &= \sum_{s=1}^{S} \int_{0}^{T} 2 w_{0} \nabla \left( \frac{1}{\rho} \nabla w_{0} \right) dt
\end{align*} \]
We then solve the normal equations by an iterative algorithm using Chebychev polynomials (cf [1]).

3 Computation of the gradient

We assume now that the normal equations have been solved exactly, and therefore we have \( r_{p} \) and \( r_{c} \) as functions of \( \rho \) and \( c \). Therefore

\[ \begin{align*}
\tilde{J}(\rho, c) &= J_{DS}(\rho, c, r_{p}(\rho, c), r_{c}(\rho, c)) \\
&= \frac{1}{2} \left\{ \| F(\rho, c, r_{p}(\rho, c), r_{c}(\rho, c)) - F_{\text{data}}\|_{L^2(\Omega; T)}^2 + \sigma_{p} \left\| \frac{\partial r_{p}(\rho, c)}{\partial x_{s}} \right\|_{L^2(\Omega)}^2 \\
&\quad + \sigma_{c} \left\| \frac{\partial r_{c}(\rho, c)}{\partial x_{s}} \right\|_{L^2(\Omega)}^2 + \lambda_{p}^2 \| W_{r_{p}}(\rho, c) \|_{L^2(\Omega)}^2 + \lambda_{c}^2 \| W_{r_{c}}(\rho, c) \|_{L^2(\Omega)}^2 \right\}
\end{align*} \]

The first derivative of \( \tilde{J} \) due to a perturbation \( (\delta \rho, \delta c) \) in \( (\rho, c) \) is given by:

\[ \begin{align*}
D\tilde{J}(\rho, c). (\delta \rho, \delta c) &= \left. \frac{\partial J_{DS}(\rho, c, r_{p}(\rho, c), r_{c}(\rho, c))}{\partial x_{s}} \right|_{(\rho, c)} \delta \rho + \frac{\partial J_{DS}(\rho, c, r_{p}(\rho, c), r_{c}(\rho, c))}{\partial c} \delta c
\end{align*} \]
Calculation of the Gradient for the DSO Objective Function

\[
+ \ D_{r_p} J_{DS}(\rho, c, r_p(\rho, c), r_c(\rho, c)) \left( D_{\rho} r_p(\rho, c) \cdot \delta \rho + D_c r_p(\rho, c) \cdot \delta c \right) \\
+ \ D_{r_c} J_{DS}(\rho, c, r_p(\rho, c), r_c(\rho, c)) \left( D_{\rho} r_c(\rho, c) \cdot \delta \rho + D_c r_c(\rho, c) \cdot \delta c \right)
\]

But since we assumed that the normal equations have been solved exactly, cf (7) then:

\[
\begin{cases}
D_{r_p} J_{DS}(\rho, c, r_p(\rho, c), r_c(\rho, c)) = 0 \\
D_{r_c} J_{DS}(\rho, c, r_p(\rho, c), r_c(\rho, c)) = 0
\end{cases}
\]

and we get the following simpler expression for the derivative of \( \bar{J} \):

\[
D \bar{J}(\rho, c). (\delta \rho, \delta c) = D_{\rho} J_{DS}(\rho, c, r_p(\rho, c), r_c(\rho, c)) \cdot \delta \rho + D_c J_{DS}(\rho, c, r_p(\rho, c), r_c(\rho, c)) \cdot \delta c
\]

Since

\[
\bar{J}(\rho, c) = \frac{1}{2} || F(\rho, c, r_p, r_c) - F_{data} ||_{L^2(0:T)}^2 + \xi(r_p, r_c)
\]

where \( \xi \) does not depend explicitly on \( \rho \) and \( c \), we have:

\[
\begin{cases}
D_{\rho} J_{DS}(\rho, c, r_p, r_c). \delta \rho = (D_{\rho} F(\rho, c, r_p, r_c). \delta \rho, F(\rho, c, r_p, r_c) - F_{data})_{L^2(0:T)} \\
D_c J_{DS}(\rho, c, r_p, r_c). \delta c = (D_c F(\rho, c, r_p, r_c). \delta c, F(\rho, c, r_p, r_c) - F_{data})_{L^2(0:T)}
\end{cases}
\]

Now to compute the gradients of \( \bar{J} \) with respect to \( \rho \) and \( c \), we must find two bilinear forms \( B_{\rho} \) and \( B_c \) such that

\[
(D_{\rho} F(\rho, c, r_p, r_c). \delta \rho, F(\rho, c, r_p, r_c) - F_{data})_{L^2(0:T)} = (\delta \rho, B_{\rho}(F(\rho, c, r_p, r_c) - F_{data}))_{L^2(\Omega)}
\]

\[
(D_c F(\rho, c, r_p, r_c). \delta c, F(\rho, c, r_p, r_c) - F_{data})_{L^2(0:T)} = (\delta c, B_c(F(\rho, c, r_p, r_c) - F_{data}))_{L^2(\Omega)}
\]

**Remark**

\( r_p \) and \( r_c \) being chosen as the solution of the normal equations, they are fixed. To enhance this fact, we use a different notation and from now on we will use \( q_p \) for \( r_p \) and \( q_c \) for \( r_c \).
We know that \( u = F(\rho, c, q, \eta_c) \) is the solution of:

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u \right) &= (q_{p} + 2q_{c}) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{q_{p}}{\rho} \nabla u_0 \right) \\
\frac{1}{\rho c^2} \frac{\partial^2 u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u_0 \right) &= f(x, z, t; x, z, z) \\
u(x, z, 0) = u_0(x, z, 0) &= \frac{\partial u}{\partial t}(x, z, 0) = \frac{\partial u_0}{\partial t}(x, z, 0) = 0 \\
u_0(0, z, t) = u_0(x, 0, t) = u_0(X, z, t) = u_0(x, Z, t) = 0 \\
u(0, z, t) = u(x, 0, t) = u(X, z, t) = u(x, Z, t) = 0
\end{align*}
\]

(13)

With a little algebra it is easy to see that \( \delta u = D_{\rho} F(\rho, c, q, \eta_c) \). \( \delta \rho \) is the solution of:

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u \right) &= -\nabla \left( \frac{\delta \rho}{\rho^2} \nabla u \right) \\
&+ \frac{\delta \rho}{\rho} \left\{ \nabla \left( \frac{1}{\rho} \nabla u \right) + (q_{p} + 2q_{c}) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{q_{p}}{\rho} \nabla u_0 \right) \right\} \\
&- (q_{p} + 2q_{c}) \nabla \left( \frac{\delta \rho}{\rho^2} \nabla u_0 \right) + \nabla \left( \frac{q_{p} \delta \rho}{\rho} \nabla u_0 \right) \\
&+ (q_{p} + 2q_{c}) \nabla \left( \frac{1}{\rho} \nabla \delta u_0 \right) - \nabla \left( \frac{q_{p}}{\rho} \nabla \delta u_0 \right)
\end{align*}
\]

(14)

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 \delta u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u_0 \right) &= \frac{\delta \rho}{\rho} \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{\delta \rho}{\rho^2} \nabla u_0 \right) \\
\delta u(x, z, 0) = \delta u_0(x, z, 0) &= \frac{\partial \delta u}{\partial t}(x, z, 0) = \frac{\partial \delta u_0}{\partial t}(x, z, 0) = 0 \\
\delta u_0(0, z, t) = \delta u_0(x, 0, t) = \delta u_0(X, z, t) = \delta u_0(x, Z, t) = 0 \\
\delta u(0, z, t) = \delta u(x, 0, t) = \delta u(X, z, t) = \delta u(x, Z, t) = 0
\end{align*}
\]
Calculation of the Gradient for the DSO Objective Function

Given a seismogram \( \varphi \), we can now evaluate the following quantity:

\[
(D_{\rho} F(\rho, c, r_{\rho}, r_{c}) \cdot \delta \rho, \varphi)_{L^2(0; T)} = \sum_{s=1}^{S} \sum_{r=1}^{R} (\varphi(x_{r}, r_{r}; x_{s}, z_{s}), \delta u(x_{r}, z_{s}; x_{s}, z_{s}))_{L^2(0; T)}
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \sum_{r=1}^{R} \varphi(x_{r}, z_{r}, t; x_{s}, z_{s}) \delta u(x_{r}, z_{s}; x_{s}, z_{s}) \delta(x - x_{s}, z - z_{s}) \ dx \ dz \ dt
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \frac{1}{\rho c^2} \frac{\partial^2 w_0}{\partial t^2} - \nabla(\frac{1}{\rho} \nabla w_0) \right) \delta u(x_{r}, z_{s}; x_{s}, z_{s}) \ dx \ dz \ dt
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} w_0(x_{r}, z_{s}; x_{s}, z_{s}) \left( \frac{1}{\rho c^2} \frac{\partial^2 u}{\partial t^2} - \nabla(\frac{1}{\rho} \nabla u) \right) \ dx \ dz \ dt
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \frac{w_0}{\rho} \left( \nabla(\frac{1}{\rho} \nabla u) + (\gamma_{\rho} + 2 q_{c}) \nabla(\frac{1}{\rho} \nabla w_0) - \nabla(\frac{\eta_{\rho}}{\rho} \nabla w_0) \right) \delta \rho \ dx \ dz \ dt
\]

\[
+ \int_{0}^{T} \int_{\Omega} \frac{1}{\rho^2} (\nabla w_0 \nabla u + \nabla((\gamma_{\rho} + 2 q_{c}) w_0) \nabla w_0 - q_{\rho} \nabla w_0 \nabla w_0) \delta \rho \ dx \ dz \ dt
\]

\[
+ \int_{0}^{T} \int_{\Omega} \left( \nabla(\frac{1}{\rho} \nabla (\gamma_{\rho} + 2 q_{c}) w_0) - \nabla(\frac{\eta_{\rho}}{\rho} \nabla w_0) \right) \delta w_0 \ dx \ dz \ dt
\]

where \( w_0 \) is the solution of (10). To find the expression of the gradient of \( \bar{J} \) with respect to \( \rho \), we need to work on the last integral. We introduce another adjoint state \( w \) defined by:

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 w}{\partial t^2} - \nabla(\frac{1}{\rho} \nabla w) &= \nabla(\frac{1}{\rho} \nabla (\gamma_{\rho} + 2 q_{c}) w_0) - \nabla(\frac{\eta_{\rho}}{\rho} \nabla w_0) \\
 w(x, z, T) &= \frac{\partial w}{\partial t}(x, z, T) = 0 \\
 w(0, z, t) &= w(X, z, t) = w(x, 0, t) = w(x, Z, t) = 0
\end{align*}
\]
We can then pursue the previous calculation:

\[
I = \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \nabla \left( \frac{1}{\rho} \nabla \left( \eta_{p} + 2\eta_{c} \right) w_{0} \right) - \nabla \left( \frac{\eta_{p}}{\rho} \nabla w_{0} \right) \right) \delta u_{0} \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \frac{1}{\rho c^{2}} \frac{\partial^{2} w_{0}}{\partial t^{2}} - \nabla \left( \frac{1}{\rho} \nabla w_{0} \right) \right) \delta u_{0} \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} w \left( \frac{1}{\rho c^{2}} \frac{\partial^{2} \delta u_{0}}{\partial t^{2}} - \nabla \left( \frac{1}{\rho} \nabla \delta u_{0} \right) \right) \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_{\Omega} \left( \int_{0}^{T} \frac{w}{\rho} \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) + \frac{1}{\rho^{2}} \nabla w \nabla u_{0} \, dt \right) \delta \rho \, dx \, dz
\]

whence:

\[
\nabla_{\rho} \bar{J} = \sum_{s=1}^{S} \int_{0}^{T} \frac{u_{0}}{\rho} \left( \nabla \left( \frac{1}{\rho} \nabla u \right) + \left( \eta_{p} + 2\eta_{c} \right) \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) - \nabla \left( \frac{\eta_{p}}{\rho} \nabla u_{0} \right) \right) \, dt
\]

\[
+ \sum_{s=1}^{S} \int_{0}^{T} \frac{1}{\rho^{2}} \left( \nabla w_{0} \nabla u + \nabla \left( \left( \eta_{p} + 2\eta_{c} \right) w_{0} \right) \nabla u_{0} - \eta_{p} \nabla w_{0} \nabla u_{0} \right) \, dt
\]

\[
+ \sum_{s=1}^{S} \int_{0}^{T} \left( \frac{u}{\rho} \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) + \frac{1}{\rho^{2}} \nabla \nabla u_{0} \left( \frac{w}{\rho} \nabla \nabla u_{0} \right) \right) \, dt
\]

(16)
Calculation of the Gradient for the DSO Objective Function

From (13) we can derive the equation verified by $\delta u = D_{c}\mathcal{F}(\rho, c, q_{\rho}, q_{c})\cdot \delta c$:

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 \delta u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u \right) &= 2 \frac{\delta c}{c} \nabla \left( \frac{1}{\rho} \nabla u \right) \\
&+ 2 \frac{\delta c}{c} \left\{ (q_{\rho} + 2q_{c}) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{q_{\rho}}{\rho} \nabla u_0 \right) \right\} \\
&+ (q_{\rho} + 2q_{c}) \nabla \left( \frac{1}{\rho} \nabla \delta u_0 \right) - \nabla \left( \frac{q_{\rho}}{\rho} \nabla \delta u_0 \right)
\end{align*}
\]

(17)

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 \delta u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u_0 \right) &= 2 \frac{\delta c}{c} \nabla \left( \frac{1}{\rho} \nabla u_0 \right) \\
\delta u(x, z, 0) &= \delta u_0(x, z, 0) = \frac{\partial \delta u}{\partial t}(x, z, 0) = \frac{\partial \delta u_0}{\partial t}(x, z, 0) = 0 \\
\delta u_0(0, z, t) &= \delta u_0(x, 0, t) = \delta u_0(X, z, t) = \delta u_0(x, Z, t) = 0 \\
\delta u(0, z, t) &= \delta u(x, 0, t) = \delta u(X, z, t) = \delta u(x, Z, t) = 0
\end{align*}
\]

Given a seismogram $\varphi$, we can now evaluate the following quantity:

\[
(D_{c}\mathcal{F}(\rho, c, q_{\rho}, q_{c})\cdot \delta c, \varphi)_{L^2(0; T)} = \sum_{s=1}^{S} \sum_{r=1}^{R} (\varphi(x_r, z_r; x_s, z_s), \delta u(x_r, z_r; x_s, z_s))_{L^2(0; T)}
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \sum_{r=1}^{R} \varphi(x_r, z_r; t; x_s, z_s) \delta u(x, z; t; x_s, z_s) \delta(x - x_s, z - z_s) \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \frac{1}{\rho c^2} \frac{\partial^2 u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u_0 \right) \right) \delta u(x, z; t; x_s, z_s) \, dx \, dz \, dt
\]

\[
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} w_0(x, z; t; x_s, z_s) \left( \frac{1}{\rho c^2} \frac{\partial^2 u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla \delta u \right) \right) \, dx \, dz \, dt
\]
\[
\sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} 2 \frac{w_{0}}{c} \left( \nabla \left( \frac{1}{\rho} \nabla u \right) + (q_{\rho} + 2q_{c}) \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) - \nabla \left( \frac{q_{\rho}}{\rho} \nabla u_{0} \right) \right) \delta c \, dx \, dz \, dt \\
+ \int_{0}^{T} \int_{\Omega} w_{0} (q_{\rho} + 2q_{c}) \nabla w_{0} \nabla \left( \frac{1}{\rho} \nabla \delta u_{0} \right) - \nabla \left( \frac{q_{\rho}}{\rho} \nabla \delta u_{0} \right) \, dx \, dz \, dt \\
= \sum_{s=1}^{S} \int_{0}^{T_{s}} \int_{\Omega} 2 \frac{w_{0}}{c} \left( \nabla \left( \frac{1}{\rho} \nabla u \right) + (q_{\rho} + 2q_{c}) \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) - \nabla \left( \frac{q_{\rho}}{\rho} \nabla u_{0} \right) \right) \delta c \, dx \, dz \, dt \\
+ \int_{0}^{T} \int_{\Omega} \delta u_{0} \left( \nabla \left( \frac{1}{\rho} \nabla (q_{\rho} + 2q_{c}) u_{0} \right) - \nabla \left( \frac{q_{\rho}}{\rho} \nabla u_{0} \right) \right) \, dx \, dz \, dt
\]

where \( w_{0} \) is the solution of (10). Introducing \( w \) the solution of (15), we can write the second integral as :

\[
\sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \nabla \left( \frac{1}{\rho} \nabla (q_{\rho} + 2q_{c}) u_{0} \right) - \nabla \left( \frac{q_{\rho}}{\rho} \nabla u_{0} \right) \right) \delta u_{0} \, dx \, dz \, dt \\
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} \left( \frac{1}{\rho c^{2}} \frac{\partial^{2} w_{0}}{\partial t^{2}} - \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) \right) \delta u_{0} \, dx \, dz \, dt \\
= \sum_{s=1}^{S} \int_{0}^{T} \int_{\Omega} w \left( \frac{1}{\rho c^{2}} \frac{\partial^{2} \delta u_{0}}{\partial t^{2}} - \nabla \left( \frac{1}{\rho} \nabla \delta u_{0} \right) \right) \, dx \, dz \, dt \\
= \sum_{s=1}^{S} \int_{\Omega} \left( \int_{0}^{T} 2 \frac{w}{c} \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) \, dt \right) \delta c \, dx \, dz
\]

whence :

\[
\left\{ \begin{array}{l}
\nabla \cdot \mathbf{J} = \sum_{s=1}^{S} \int_{0}^{T} 2 \frac{w_{0}}{c} \left( \nabla \left( \frac{1}{\rho} \nabla u \right) + (q_{\rho} + 2q_{c}) \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) - 2 \frac{w_{0}}{c} \nabla \left( \frac{q_{\rho}}{\rho} \nabla u_{0} \right) \right) \, dt \\
+ \sum_{s=1}^{S} \int_{0}^{T} 2 \frac{w}{c} \nabla \left( \frac{1}{\rho} \nabla u_{0} \right) \, dt
\end{array} \right.
\]

(18)
4 Implementation of the Gradient

We now turn to the implementation aspects of the computation of the two gradients obtained above. We are going to use a finite difference method to compute the different wave fields we need. We summarize below the expressions of the two gradients and the equations we need to discretize to compute them.

\[
\nabla_\rho \tilde{J} = \sum_{s=1}^{S} \int_{0}^{T} \frac{w_0}{\rho} \left( \nabla \left( \frac{1}{\rho} \nabla u \right) + \left( q_\rho + 2q_c \right) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{q_\rho}{\rho} \nabla u_0 \right) \right) \, dt
\]

\[
+ \sum_{s=1}^{S} \int_{0}^{T} \frac{1}{\rho^2} \left( \nabla w_0 \nabla u + \nabla \left( (q_\rho + 2q_c) w_0 \right) \nabla u_0 - q_\rho \nabla w_0 \nabla u_0 \right) \, dt
\]

\[
+ \sum_{s=1}^{S} \int_{0}^{T} \left( \frac{w}{\rho} \nabla \left( \frac{1}{\rho} \nabla u_0 \right) + \frac{1}{\rho^2} \nabla w \nabla u_0 \right) \, dt
\]

\[
\nabla_c \tilde{J} = \sum_{s=1}^{S} \int_{0}^{T} 2 \frac{u_0}{c} \left( \nabla \left( \frac{1}{\rho} \nabla u \right) + \left( q_\rho + 2q_c \right) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - 2 \frac{u_0}{c} \nabla \left( \frac{q_\rho}{\rho} \nabla u_0 \right) \right) \, dt
\]

\[
+ \sum_{s=1}^{S} \int_{0}^{T} 2 \frac{u}{c} \nabla \left( \frac{1}{\rho} \nabla u_0 \right) \, dt
\]

where the two direct states \( u_0, u \) are solutions of:

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 u_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u_0 \right) &= f(x, z, t; x_s, z_s) \\
\frac{\partial u_0}{\partial t}(x, z, 0) &= 0 \\
u_0(x, z, 0) &= 0 \\
u_0(x, 0, t) &= 0 \\
u_0(0, z, t) &= 0 \\
u_0(x, z, t) &= 0 \\
\frac{1}{\rho c^2} \frac{\partial^2 u}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla u \right) &= \left( q_\rho + 2q_c \right) \nabla \left( \frac{1}{\rho} \nabla u_0 \right) - \nabla \left( \frac{q_\rho}{\rho} \nabla u_0 \right) \\
u(x, z, 0) &= 0 \\
u(x, z, 0) &= 0 \\
u(0, z, t) &= 0 \\
u(x, 0, t) &= 0 \\
u(x, z, t) &= 0
\end{align*}
\]
and the two adjoint states $w_0$ and $w$ are solutions of:

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 w_0}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla w_0 \right) &= \sum_{r=1}^{R} (F(\rho, c, q_{\mu}, q_{\lambda})(x_r, z_r, t; x_t, z_t) - F_{data}(t)) \delta(x - x_r, z - z_r) \\
w_0(x, z, T) &= \frac{\partial w_0}{\partial t}(x, z, T) = 0 \\
w_0(0, z, t) = w_0(X, z, t) = w_0(x, 0, t) = w_0(x, Z, t) = 0
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\rho c^2} \frac{\partial^2 w}{\partial t^2} - \nabla \left( \frac{1}{\rho} \nabla w \right) &= \nabla \left( \frac{1}{\rho} \nabla (q_{\mu} + 2q_{\lambda}) w_0 \right) - \nabla \left( \frac{q_{\mu}}{\rho} \nabla w_0 \right) \\
w(x, z, T) &= \frac{\partial w}{\partial t}(x, z, T) = 0 \\
w(0, z, t) = w(X, z, t) = w(x, 0, t) = w(x, Z, t) = 0
\end{align*}
\]

The finite difference schemes used to simulate those wave fields are of order 2 in time and 2L in space, where L is the number of points used in the calculation of the derivative. Those schemes are described in detail in [4].

In the previous calculations, we repeatedly used the vanishing of $u_0$, $u$, $w_0$ and $w$ on the boundary of the domain. In order for these calculations to carry over to the discretized equations we need to make sure that $u_{0,h}$, $u_h$, $w_{0,h}$ and $w_h$, discrete equivalents of $u_0$, $u$, $w_0$ and $w$, have the same property, with the same consequences.

For this we use the well known 'image principle', by extending $u_{0,h}$, $u_h$, $w_{0,h}$ and $w_h$ outside the domain.

First, we define some functional spaces, to which $u_{0,h}$, $u_h$, $w_{0,h}$ and $w_h$ will belong.

\[
L^2_{\sigma,}\sigma = \left\{ \varphi \in L^2(\Omega) \mid \varphi = \sum_{i=1}^{N_x} \sum_{j=1}^{N_x} \varphi_{i,j} 1_{[(i-\frac{1}{2})\Delta x,(i+\frac{1}{2})\Delta x] \times [(j-\frac{1}{2})\Delta z,(j+\frac{1}{2})\Delta z]}(x, z) \right\}
\]

\[
L^2_{*,*} = \left\{ \varphi \in L^2(\Omega) \mid \varphi = \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_x-1} \varphi_{i+\frac{1}{2},j+\frac{1}{2}} 1_{[i\Delta x,(i+1)\Delta x] \times [j\Delta z,(j+1)\Delta z]}(x, z) \right\}
\]
Calculation of the Gradient for the DSO Objective Function

\[ L_{2,0}^2 = \left\{ \varphi \in L^2(\Omega) \mid \varphi = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \varphi_{i,j} \cdot 1_{[(i-\frac{1}{2})\Delta x, (i+\frac{1}{2})\Delta x] \times [(j-\frac{1}{2})\Delta z, (j+\frac{1}{2})\Delta z]}(x, z) \right\} \]

\[ L_{2,0}^2 = \left\{ \varphi \in L^2(\Omega) \mid \varphi = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \varphi_{i,j} \cdot 1_{[(i-\frac{1}{2})\Delta x, (i+\frac{1}{2})\Delta x] \times [(j-\frac{1}{2})\Delta z, (j+\frac{1}{2})\Delta z]}(x, z) \right\} \]

where

\[ 1_{[a,b] \times [c,d]}(x, z) = \begin{cases} 1 & (x, z) \in [a, b] \times [c, d] \\ 0 & (x, z) \not\in [a, b] \times [c, d] \end{cases} \]

We approximate the first derivative by the following operator:

\[ A_x^2 : L_{2,0}^2 \rightarrow L_{2,0}^2 \]

\[ u \rightarrow A_x^2 u(i + \frac{1}{2}, j) = \sum_{l=1}^{L} \frac{\beta_l}{\Delta x} [u(i + l, j) - u(i - l + 1, j)] \]

\( A_x^2 \) is a finite difference approximation of order 2L in \(((i+\frac{1}{2})\Delta x, j\Delta z)\) of the quantity \( \frac{\partial u}{\partial x} \), with the coefficients \( \beta_l \) defined in Appendix 1. The exponent refers to the departure set \( L_{2,0}^2 \); the subscript to the direction of differentiation. Similarly we define:

\[ A_x^2 : L_{2,0}^2 \rightarrow L_{2,0}^2 \]

\[ u \rightarrow A_x^2 u(i, j + \frac{1}{2}) = \sum_{l=1}^{L} \frac{\beta_l}{\Delta x} [u(i, j + l) - u(i, j - l + 1)] \]

\[ A_x^2 : L_{2,0}^2 \rightarrow L_{2,0}^2 \]

\[ v \rightarrow A_x^2 v(i, j + \frac{1}{2}) = \sum_{l=1}^{L} \frac{\beta_l}{\Delta x} [v(i + l + \frac{1}{2}, j + \frac{1}{2}) - v(i - l + \frac{1}{2}, j + \frac{1}{2})] \]

\[ A_x^2 : L_{2,0}^2 \rightarrow L_{2,0}^2 \]

\[ v \rightarrow A_x^2 v(i, j + \frac{1}{2}) = \sum_{l=1}^{L} \frac{\beta_l}{\Delta x} [v(i + \frac{1}{2}, j + l + \frac{1}{2}) - v(i + \frac{1}{2}, j - l + \frac{1}{2})] \]

We approximate the quantity \( \nabla \left( \frac{1}{\rho} \nabla u \right) \) by \( \nabla_h \left( \frac{1}{\rho} \nabla_h u \right) = -^t A_x^2 \left( -^t A_x^2 u \right) - ^t A_x^2 \left( -^t A_x^2 u \right) \)
Fig 1: The original and shifted grids

The finite difference operators use $2L$ points to compute the derivative. For instance, $A_x^2 u$ approximates $\frac{\partial u}{\partial x}$ in $((i + \frac{1}{2}) \Delta x, j \Delta z)$ with $L$ points to the right, and $L$ points to the left of $((i + \frac{1}{2}) \Delta x, j \Delta z)$.

Therefore when we reach the boundary of the domain, for instance, we miss points to compute our derivative. A way to solve that problem is to extend the function we need outside the domain $\Omega$. There are many ways to do so, but we keep in mind that we want our discretized functions to be subject to Dirichlet boundary conditions.

We are going to derive the way to extend the discretized functions, so that they will vanish on the boundary, and the operators $\nabla_h (\frac{1}{\rho} \nabla_h u)$ will be self adjoint.

Since the derivatives are taken, each time in one direction, it is equivalent to consider a unidimensional problem. We consider $\Omega = [0, X]$ and we set $\Omega_o = [1, 2.. N]$ and $\Omega_e = [1, 2..N - 1]$. We define the finite difference operators as follows:

\[ A^o : L^2(\Omega_o) \rightarrow L^2(\Omega_o) \]
\[ u \rightarrow A^o u(i + \frac{1}{2}) = \sum_{i=1}^{L} a_i [u_{i+1} - u_{i-1+1}] \]

\[ A^* : L^2(\Omega_e) \rightarrow L^2(\Omega_e) \]
\[ v \rightarrow A^* v(i) = \sum_{i=1}^{L} a_i [u_{i+1- \frac{1}{2}} - u_{i-1+ \frac{1}{2}}] \]
Calculation of the Gradient for the DSO Objective Function

with \( a_i = \beta_i/\Delta x \) and:

\[
L^2(\Omega_o) = \left\{ \varphi \in L^2(\Omega) / \varphi = \sum_{i=1}^{N_x} \varphi_i 1_{[(i-\frac{1}{2})\Delta x,(i+\frac{1}{2})\Delta x]}(z) \right\}
\]

\[
L^2(\Omega_\ast) = \left\{ \varphi \in L^2(\Omega) / \varphi = \sum_{i=1}^{N_x-1} \varphi_{i+\frac{1}{2}} 1_{[i\Delta x,(i+1)\Delta x]} \right\}
\]

Those spaces are provided with the usual scalar products defined by:

\[
(f, g)_{L^2(\Omega_o)} = (f, g)_o = \sum_{i=1}^{N} f_i g_i \Delta x \Delta z
\]

\[
(f, g)_{L^2(\Omega_\ast)} = (f, g)_\ast = \sum_{i=1}^{N_x-1} f_{i+\frac{1}{2}} g_{i+\frac{1}{2}} \Delta x \Delta z
\]

We suppose that the boundaries of the domain are located in \( i = 1 \) and \( i = N \), so that \( u_1 = u_N = 0 \). We extend \( u \in \Omega_o \), outside \( \Omega_o \) by the following procedure:

\[
u_{1-k} = -u_{1+k} \quad k = 0..L - 1
\]

\[
u_{N+k} = -u_{N-k} \quad k = 0..L - 1
\]

That is we skew-symetrize \( u \), at the boundary. Now we want to find under what conditions we have

\[
(Au, v)_o = (v, {}^t Au)_o
\]
That is an integration by part without boundary terms. We have:

\[(Au, v)_o = \sum_{i=1}^{N-1} Au_{i+\frac{1}{2}} v_{i+\frac{1}{2}}\]

\[= \sum_{i=1}^{N-1} \sum_{l=1}^{L} a_l (u_{i+l} - u_{i-l+1}) v_{i+\frac{1}{2}}\]

\[= \sum_{l=1}^{L} a_l \left( \sum_{i=1}^{N-1} u_{i+l} v_{i+\frac{1}{2}} - \sum_{i=1}^{N-1} u_{i-l+1} v_{i+\frac{1}{2}} \right)\]

\[= \sum_{l=1}^{L} a_l \left( \sum_{j=l+1}^{N-l} u_j v_{j-l-\frac{1}{2}} - \sum_{j=1-l}^{N-l} u_j v_{j+l-\frac{1}{2}} \right)\]

\[= \sum_{i=1}^{N} \sum_{l=1}^{L} a_l (v_{j-l+\frac{1}{2}} - v_{j+l-\frac{1}{2}}) u_j - \sum_{l=1}^{L} a_l \left( \sum_{j=1}^{l} u_j v_{j-l+\frac{1}{2}} + \sum_{j=2-l}^{0} u_j v_{j+l-\frac{1}{2}} \right)\]

\[+ \sum_{l=1}^{L} a_l \left( \sum_{j=N-l+1}^{N} u_j v_{j+l-\frac{1}{2}} - \sum_{j=1-l}^{N-l} u_j v_{j+l-\frac{1}{2}} \right)\]

\[= (v, 'Au)_o - \sum_{l=1}^{L} a_l \left( \sum_{j=1}^{l} u_j v_{j-l+\frac{1}{2}} + \sum_{j=2-l}^{0} u_j v_{j+l-\frac{1}{2}} \right)\]

\[+ \sum_{l=1}^{L} a_l \left( \sum_{j=N-l+1}^{N} u_j v_{j+l-\frac{1}{2}} - \sum_{j=N-l+1}^{N-l} u_j v_{j+l-\frac{1}{2}} \right)\]

Therefore we have boundary terms, given by

\[B_1 = \sum_{l=1}^{L} a_l \left( \sum_{j=1}^{l} u_j v_{j-l+\frac{1}{2}} + \sum_{j=2-l}^{0} u_j v_{j+l-\frac{1}{2}} \right)\]

\[B_2 = \sum_{l=1}^{L} a_l \left( \sum_{j=N-l+1}^{N} u_j v_{j+l-\frac{1}{2}} + \sum_{j=N-l+1}^{N-l} u_j v_{j+l-\frac{1}{2}} \right)\]
Calculation of the Gradient for the DSO Objective Function

We have

\[ B_1 = \sum_{l=1}^{L} a_l \left( \sum_{j=1}^{l} u_j v_{j-l+\frac{1}{2}} + \sum_{j=l+1}^{0} u_j v_{j+l-\frac{1}{2}} \right) \]

\[ = \sum_{l=1}^{L} a_l \left( \sum_{j=2}^{l} u_j v_{j-l+\frac{1}{2}} + \sum_{k=1}^{l-1} u_{1-k} v_{-k+l+\frac{1}{2}} \right) \]

\[ = \sum_{l=1}^{L} a_l \left( \sum_{k=1}^{l-1} u_{1+k} v_{j-l+\frac{1}{2}} + u_{1-k} v_{-k+l+\frac{1}{2}} \right) \]

\[ = \sum_{l=1}^{L} a_l \left( \sum_{k=1}^{l-1} u_{1+k} (v_{j-l+\frac{1}{2}} - v_{-k+l+\frac{1}{2}}) \right) \]

and

\[ B_2 = \sum_{l=1}^{L} a_l \left( \sum_{j=N+1}^{N+l-1} u_j v_{j-l+\frac{1}{2}} + \sum_{j=N-l+1}^{N} u_j v_{j+l-\frac{1}{2}} \right) \]

\[ = \sum_{l=1}^{L} a_l \left( \sum_{k=1}^{l} u_{N+k} v_{N+k-l+\frac{1}{2}} + \sum_{k=0}^{l-1} u_{N-k} v_{N-k+l-\frac{1}{2}} \right) \]

\[ = \sum_{l=1}^{L} a_l \left( \sum_{k=1}^{l} u_{N+k} v_{N+k-l+\frac{1}{2}} + \sum_{k=1}^{l-1} u_{N-k} v_{N-k+l-\frac{1}{2}} \right) \]

\[ = \sum_{l=1}^{L} a_l \left( \sum_{k=1}^{l-1} u_{N-k} (v_{N-k+l-\frac{1}{2}} - v_{N+k-l+\frac{1}{2}}) \right) \]

We see that if we extend \( v \) outside \( \Omega_* \) by symmetry, that is:

\[ v_{-k+\frac{1}{2}} = v_{k+\frac{1}{2}} \quad k = 0..L - 1 \]

\[ v_{N+\frac{1}{2}+k} = v_{N+\frac{1}{2}-k} \quad k = 0..L - 1 \]

we annihilate the boundary terms, \( B_1 \) and \( B_2 \).
To summarize, with:

\[ u_{1-k} = -u_{1+k} \quad k = 0..L - 1 \]

\[ u_{N+k} = -u_{N-k} \quad k = 0..L - 1 \]

\[ v_{-k+\frac{1}{2}} = v_{k+\frac{1}{2}} \quad k = 0..L - 1 \]

\[ v_{N+\frac{1}{2}+k} = v_{N+\frac{1}{2}-k} \quad k = 0..L - 1 \]

the implementation of \( \nabla \left( \frac{1}{\rho} \nabla u \right) \) by \( \nabla_h \left( \frac{1}{\rho} \nabla_h u_{0,h} \right) = -\frac{\rho}{\rho} A^{\ast}\left( \frac{1}{\rho} A_{x}^{\ast} u \right) - \frac{\rho}{\rho} A_{x}^{\ast}\left( \frac{1}{\rho} A_{y}^{\ast} u \right) \) defines an self-adjoint operator, and the integrations by parts carried out in the previous section for the computation of the gradients, carry over to the discrete case. Therefore we define \( u_{0,h}, u_{h} \) as solutions of:

\[
\left\{ \begin{array}{l}
\frac{1}{\rho c^2} \frac{\partial^2 u_{0,h}}{\partial t^2} - \nabla_h \left( \frac{1}{\rho} \nabla_h u_{0,h} \right) = f(x, z, t; x_2, z_2) \\
u_{0,h}(x, z, 0) = \frac{\partial u_{0,h}}{\partial t}(x, z, 0) = 0 \\
u_{0,h}(0, z, t) = u_{0,h}(X, z, t) = u_{0,h}(x, 0, t) = u_{0,h}(x, Z, t) = 0
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
\frac{1}{\rho c^2} \frac{\partial^2 u_{h}}{\partial t^2} - \nabla_h \left( \frac{1}{\rho} \nabla_h u_{h} \right) = \left( q_{\rho} + 2q_{c} \right) \nabla_h \left( \frac{1}{\rho} \nabla_h u_{0,h} \right) - \nabla_h \left( \frac{q_{c}}{\rho} \nabla_h u_{0,h} \right) \\
u_{h}(x, z, 0) = \frac{\partial u_{h}}{\partial t}(x, z, 0) = 0 \\
u_{h}(0, z, t) = u_{h}(x, z, t) = u_{h}(x, 0, t) = u_{h}(x, Z, t) = 0
\end{array} \right.
\]

and the two adjoint states \( u_{0,h} \) and \( u_{h} \) as solutions of:
Calculation of the Gradient for the DSO Objective Function

\[
\left\{ \begin{array}{l}
\frac{1}{\rho c^2} \frac{\partial^2 w_{0,h}}{\partial t^2} - \nabla_h \left( \frac{1}{\rho} \nabla_h w_{0,h} \right) = \sum_{r=1}^{R} \left( F(\rho, c, q_r, q_c)(x_r, z_r, t; x_z, z_z) - F_{data}(t) \right) \delta(x - x_z, z - z_z)

w_{0,h}(x, z, T) = \frac{\partial w_{0,h}}{\partial t}(x, z, T) = 0

w_{0,h}(0, z, t) = w_{0,h}(X, z, t) = w_{0,h}(x, 0, t) = w_{0,h}(x, Z, t) = 0
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
\frac{1}{\rho c^2} \frac{\partial^2 w_{h}}{\partial t^2} - \nabla_h \left( \frac{1}{\rho} \nabla_h w_{h} \right) = \nabla_h \left( \frac{1}{\rho} \nabla_h (q_r + 2q_c) w_{0,h} \right) - \nabla_h \left( \frac{q_r}{\rho} \nabla_h w_{0,h} \right)

w_{h}(x, z, T) = \frac{\partial w_{h}}{\partial t}(x, z, T) = 0

w_{h}(0, z, t) = w_{h}(X, z, t) = w_{h}(x, 0, t) = w_{h}(x, Z, t) = 0
\end{array} \right.
\]

The two gradients are now given by

\[
\nabla_{\rho,h}\tilde{J} = \sum_{s=1}^{S} \sum_{n=1}^{N} \frac{w_{0,h}}{\rho} \left( \nabla_h \left( \frac{1}{\rho} \nabla_h u_{rh} \right) + (q_r + 2q_c) \nabla_h \left( \frac{1}{\rho} \nabla_h u_{0,h} \right) - \nabla_h \left( \frac{q_r}{\rho} \nabla_h u_{0,h} \right) \right) \Delta t

+ \sum_{s=1}^{S} \sum_{n=1}^{N} \frac{1}{\rho^2} \left( \nabla_h u_{0,h} \nabla_h u_{rh} + \nabla_h \left( (q_r + 2q_c) w_{0,h} \right) \nabla_h u_{0,h} - q_r \nabla_h u_{0,h} \nabla_h u_{0,h} \right) \Delta t

+ \sum_{s=1}^{S} \sum_{n=1}^{N} \left( \frac{w_{h}}{\rho} \nabla_h \left( \frac{1}{\rho} \nabla_h u_{0,h} \right) + \frac{1}{\rho^2} \nabla_h u_{h} \nabla_h u_{0,h} \right) \Delta t
\]

\[
\nabla_{c,h}\tilde{J} = \sum_{s=1}^{S} \sum_{n=1}^{N} \frac{w_{0,h}}{c} \left( \nabla_h \left( \frac{1}{\rho} \nabla_h u_{rh} \right) + (q_r + 2q_c) \nabla_h \left( \frac{1}{\rho} \nabla_h u_{0,h} \right) - 2 \frac{w_{0,h}}{c} \nabla_h \left( \frac{q_r}{\rho} \nabla_h u_{0,h} \right) \right) \Delta t

+ \sum_{s=1}^{S} \sum_{n=1}^{N} \frac{w_{h}}{c} \nabla_h \left( \frac{1}{\rho} \nabla_h u_{0,h} \right) \Delta t
\]
Remark:

It is important to notice the different contribution of a variation in density, or a variation in velocity. The first one will essentially have an effect on the amplitude of the reflected signals, whereas the second one will have an effect on the kinematic of the different arrivals. Therefore a variation in velocity will be much more non linear effect on the cost function than a variation in density.

That is why, thinking of the computational cost of gradients, it would be interesting to drop the computation of the gradient with respect to \( \rho \), and account for the variation of amplitude in the 'inner' variable \( r_\rho \).

5 Appendix

5.1 Appendix 1

The coefficients \( \beta_i \) are defined by \( \beta_i = \alpha_i / (2l - 1) \). For consistency reasons \( \alpha_i \) are solutions of

\[
\sum_{i=1}^{L} \alpha_i = 1
\]

\[
\sum_{i=1}^{L} (2l - 1)^{2p} \alpha_i = 0 \quad p = 1..L
\]

Solving this linear system gives :

\[
\alpha_i = (-1)^{i+1} \frac{\prod_{m \neq l}^{L} (2m - 1)}{\prod_{m \neq l}^{L} \left| (2m - 1)^2 - (2l - 1)^2 \right|}
\]
Calculation of the Gradient for the DSO Objective Function

References


