Calculation and Implementation of the Gradient of the DSO Objective Function for the General Acoustic Model

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Abstract

We present in this paper the computation of the DSO objective function in the general acoustic case. In this model, the density and the velocity are functions of the space variables. We use a pertubationnal approach, justified by the separation of scales between the long and short wavelength components of the model. An extension of the adjoint state technique yields an accurate expression of the gradient of the DSO objective function. Then we use a finite difference approximation of the wave equation, and give in the discrete case the expression of the gradient. We show in that case how to apply the principle of images, so that the discrete operators involved are self-adjoint and give exact discrete integration by parts.

1 The Forward Map

We consider the following wave propagation problem. Given an acoustic medium defined by its the density $\rho(x,z)$ and its velocity $c(x,z)$ for $(x,z) \in \Omega =]0;X[\times]0;Z[\quad t \in]0;T[$, find the pressure field $u(x,z,t)$ which is the solution of the radiation problem :

$$(1) \qquad \begin{cases} \dfrac{1}{\rho c^2(x,z)} \dfrac{\partial^2 u}{\partial t^2}(x,t) - \nabla(\dfrac{1}{\rho(x,z)} \nabla u(x,t)) = f(x,z,t;x_s,z_s) \\ u(x,z,0) = \dfrac{\partial u}{\partial t}(x,z,0) = 0 \\ u(0,z,t) = u(X,z,t) = u(x,0,t) = u(x,Z,t) = 0 \end{cases}$$

here (x_s, z_s) is the location of the source. To find solutions of this problem, we adopt a pertubationnal approach. We look for solutions 'close' to a reference solution u_0 given for a reference distribution of density ρ_0 and velocity c_0. We therefore suppose ρ and c to be given by

$$(2) \qquad \begin{cases} \rho(x,z) = \rho_0(x,z) + \delta\rho(x,z) \\ c(x,z) = c_0(x,z) + \delta c(x,z) \end{cases}$$

We use in those expressions the separation of scales inherent to the physics of the problem. $c_0(x,z)$ and $\rho_0(x,z)$ are supposed to be smooth functions, and $\delta c(x,z)$ and $\delta\rho(x,z)$ are supposed to be oscillatory functions. We suppose that the pertubations are relatively small compared to the references that is :

$$||\frac{\delta\rho}{\rho_0}|| << 1 \qquad ||\frac{\delta c}{c_0}|| << 1$$

where $||.||$ is a certain norm on the functional space where ρ and c are defined (e.g the L^2 norm). We will see as the calculation proceeds the necessary regularity of the different so called models.

Then we look for a solution written as $u = u_0 + \delta u$ where u_0 verifies (1) and δu is the solution of the linearised problem :

$$(3) \qquad \begin{cases} \dfrac{1}{\rho_0 c_0^2} \dfrac{\partial^2 \delta u}{\partial t^2} - \nabla(\dfrac{1}{\rho_0} \nabla \delta u) = \dfrac{1}{\rho_0 c_0^2} \left(\dfrac{\delta\rho}{\rho_0} + 2\dfrac{\delta c}{c_0} \right) \dfrac{\partial^2 u_0}{\partial t^2} - \nabla(\dfrac{\delta\rho}{\rho_0^2} \nabla u_0) \\ \delta u(x,z,0) = \dfrac{\partial \delta u}{\partial t}(x,z,0) = 0 \\ \delta u(0,z,t) = \delta u(X,z,t) = \delta u(x,0,t) = \delta u(x,Z,t) = 0 \end{cases}$$

We define the reflectivities (cf [1]), as the relative perturbations, that is :

$$r_\rho = \frac{\delta\rho}{\rho_0} \qquad r_c = \frac{\delta c}{c_0}$$

Supposing that f, r_ρ and r_c do not have overlapping supports, we can write (3) as follows

$$(4) \qquad \begin{cases} \dfrac{1}{\rho_0 c_0^2}\dfrac{\partial^2 \delta u}{\partial t^2} - \nabla(\dfrac{1}{\rho_0}\nabla\delta u) = (r_\rho + 2r_c)\nabla(\dfrac{1}{\rho_0}\nabla u_0) - \nabla(\dfrac{r_\rho}{\rho_0}\nabla u_0) \\ \delta u(x,z,0) = \dfrac{\partial\delta u}{\partial t}(x,z,0) = 0 \\ \delta u(0,z,t) = \delta u(X,z,t) = \delta u(x,0,t) = \delta u(x,Z,t) = 0 \end{cases}$$

We are now able to define the forward map F of our inverse problem. It maps the functions defining the medium, the density ρ, the velocity c, the reflectivity in density r_ρ and the reflectivity in velocity r_c to the seismogram produced in this medium at the array of receivers $(x_r, z_r)_{r=1..R}$ by the source $f(t)$ located in $(x_s, z_s)_{s=1..S}$.
Therefore we define the forward map by :

$$F(\rho, c, r_\rho, r_c)(t; x_s, z_s) = \sum_{s=1}^{S}\sum_{r=1}^{R} u(x_r, z_r, t; x_s, z_s)$$

where u satisfies :

$$(5) \qquad \begin{cases} \dfrac{1}{\rho c^2}\dfrac{\partial^2 u_0}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla u_0) = f(x,z,t;x_s,z_s) \\ \dfrac{1}{\rho c^2}\dfrac{\partial^2 u}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla u) = (r_\rho + 2r_c)\nabla(\dfrac{1}{\rho}\nabla u_0) - \nabla(\dfrac{r_\rho}{\rho}\nabla u_0) \\ u(x,z,0) = u_0(x,z,0) = \dfrac{\partial u}{\partial t}(x,z,0) = \dfrac{\partial u_0}{\partial t}(x,z,0) = 0 \\ u_0(0,z,t) = u_0(x,0,t) = u_0(X,z,t) = u_0(x,Z,t) = 0 \\ u(0,z,t) = u(x,0,t) = u(X,z,t) = u(x,Z,t) = 0 \end{cases}$$

2 The Objective Function and the Normal Equations

Following [2], we introduce the Differential Semblance Objective function as :

$$(6)\quad \begin{cases} \tilde{J}(\rho,c) &= \min_{\tau_\rho,\tau_c} J_{DS}(\rho,c,\tau_\rho,\tau_c) \\ J_{DS}(\rho,c,\tau_\rho,\tau_c) &= \frac{1}{2}\Big\{ \|F(\rho,c,\tau_\rho,\tau_c) - F_{data}\|^2_{L^2(0;T)} + \sigma_\rho \|\frac{\partial \tau_\rho}{\partial x_s}\|^2_{L^2(\Omega)} \\ &+ \sigma_c \|\frac{\partial \tau_c}{\partial x_s}\|^2_{L^2(\Omega)} + \lambda_\rho^2 \|W\tau_\rho\|^2_{L^2(\Omega)} + \lambda_c^2 \|W\tau_c\|^2_{L^2(\Omega)} \Big\} \end{cases}$$

where W is a regularizing operator, for instance $W = I$ or $W = \nabla_{x,z}$ (cf [3]). The first term of J_{DS} fits the data and the other part of J_{DS} enforces coherency in the inverted models (cf [3]). When we use $W = \nabla_{x,z}$ we need τ_ρ and τ_c to belong to $H^1(\Omega)$.
We see by the expression of the cost function alone, that we have two minimization problems to solve.
First the minimization on the so called 'inner variables' τ_ρ and τ_c. Then once τ_ρ and τ_c determined at ρ and c fixed, we want to minimize $\tilde{J}$ over the so called 'outer variables' ρ and c.
We start with the inner variables minimization. We want to compute the gradient of J_{DS} with respect to τ_ρ and τ_c. The first variation δJ_{DS} of J_{DS} with respect to a variation $\delta\tau_\rho$ and $\delta\tau_c$ in τ_ρ and τ_c is given by :

$$\begin{aligned} \delta J_{DS} &= (D_{\tau_\rho}F.\delta\rho + D_{\tau_c}F.\delta c, F - F_{data})_{L^2(0;T)} + \sigma_\rho^2 \left(\frac{\partial \tau_\rho}{\partial x_s}, \frac{\partial \delta\tau_\rho}{\partial x_s}\right)_{L^2(\Omega)} \\ &+ +\sigma_c^2 \left(\frac{\partial \tau_c}{\partial x_s}, \frac{\partial \delta\tau_c}{\partial x_s}\right)_{L^2(\Omega)} + \lambda_\rho^2 (W\delta\tau_\rho, W\tau_\rho)_{L^2(\Omega)} + \lambda_c^2 (W\delta\tau_c, W\tau_c)_{L^2(\Omega)} \end{aligned}$$

where $D_{\tau_\rho}F$ is the derivative of J_{DS} with respect to τ_ρ and $D_{\tau_\rho}F$ is the derivative of J_{DS} with respect to τ_c. This can be written as :

$$\begin{aligned} \delta J_{DS} &= \left(\delta\rho, D_{\tau_\rho}F^*(F - F_{data}) - \sigma_\rho^2 \frac{\partial^2 \tau_\rho}{\partial x_s^2} + \lambda_\rho^2 W^T W \tau_\rho\right)_{L^2(\Omega)} \\ &+ \left(\delta c, D_{\tau_c}F^*(F - F_{data}) - \sigma_c^2 \frac{\partial^2 \tau_c}{\partial x_s^2} + \lambda_c^2 W^T W \tau_c\right)_{L^2(\Omega)} \end{aligned}$$

where $D_{\tau_\rho}F^*$ is the adjoint operator of the derivative $D_{\tau_\rho}F$ and $D_{\tau_c}F^*$ is the adjoint of the derivative $D_{r_\rho}F$. This simply means that :

$$(7) \quad \begin{cases} \nabla_{r_\rho} J_{DS} &= D_{r_\rho}F^*(F - F_{data}) - \sigma_\rho^2 \dfrac{\partial^2 r_\rho}{\partial x_s^2} + \lambda_\rho^2 W^T W r_\rho \\ \nabla_{r_c} J_{DS} &= D_{r_c}F^*(F - F_{data}) - \sigma_c^2 \dfrac{\partial^2 r_c}{\partial x_s^2} + \lambda_c^2 W^T W r_c \end{cases}$$

When we use $W = \nabla_{x,z}$, we have to suppose that r_ρ and r_c belong to $H^2(\Omega)$. Setting the gradients to zero in (7) we get the following normal equations :

$$(8) \quad \begin{cases} M_{r_\rho}F - \sigma_\rho^2 \dfrac{\partial^2 r_\rho}{\partial x_s^2} + \lambda_\rho^2 W^T W r_\rho &= M_{r_\rho}F_{data} \\ M_{r_c}F - \sigma_c^2 \dfrac{\partial^2 r_c}{\partial x_s^2} + \lambda_c^2 W^T W r_c &= M_{r_c}F_{data} \end{cases}$$

Now we need to know the effect of the operator $D_{r_\rho}F^* = M_{r_\rho}$ and $D_{r_c}F^* = M_{r_c}$ on some seismogram $\varphi(x_r, z_r, t; x_s, z_s)$. Since F is linear in r_ρ and r_c, we have :

$$\begin{cases} D_{r_\rho}F.\delta r_\rho &= F(\rho, c, \delta r_\rho, 0) \\ D_{r_c}F.\delta r_c &= F(\rho, c, 0, \delta r_c) \end{cases}$$

therefore

$$\begin{aligned} (M_{r_\rho}\varphi, \delta r_\rho)_{L^2(\Omega)} &= \sum_{s=1}^{S}\sum_{r=1}^{R} (\varphi(x_r, z_r; x_s, z_s), D_{r_\rho}F(\rho, c, r_\rho, r_c).\delta r_\rho)_{L^2(0;T)} \\ &= \sum_{s=1}^{S}\sum_{r=1}^{R} (\varphi(x_r, z_r; x_s, z_s), F(\rho, c, r_\rho, 0).\delta r_\rho)_{L^2(0;T)} \end{aligned}$$

We have $F(\rho, c, r_\rho, 0) = \delta u$ solution of :

$$(9)\quad \begin{cases} \dfrac{1}{\rho c^2}\dfrac{\partial^2 \delta u}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla\delta u) = \delta r_\rho \nabla(\dfrac{1}{\rho}\nabla u_0) - \nabla(\dfrac{\delta r_\rho}{\rho}\nabla u_0) \\ \dfrac{1}{\rho c^2}\dfrac{\partial^2 u_0}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla u_0) = f(x, z, t; x_s, z_s) \\ \delta u(x, z, 0) = \dfrac{\partial \delta u}{\partial t}(x, z, 0) = 0 \\ \delta u(0, z, t) = \delta u(X, z, t) = \delta u(x, 0, t) = \delta u(x, Z, t) = 0 \\ u_0(x, z, 0) = \dfrac{\partial u_0}{\partial t}(x, z, 0) = 0 \\ u_0(0, z, t) = u_0(X, z, t) = u_0(x, 0, t) = u_0(x, Z, t) = 0 \end{cases}$$

Following a well known technique (cf [5], [6], [7]), we define the adjoint w_0 state as follows

$$(10)\quad \begin{cases} \dfrac{1}{\rho c^2}\dfrac{\partial^2 w_0}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla w_0) = \displaystyle\sum_{r=1}^{R} \varphi(x_r, z_r, t; x_s, z_s)\delta(x - x_s, z - z_s) \\ w_0(x, z, T) = \dfrac{\partial w_0}{\partial t}(x, z, T) = 0 \\ w_0(0, z, t) = w_0(X, z, t) = w_0(x, 0, t) = w_0(x, Z, t) = 0 \end{cases}$$

then we have :

$$\begin{aligned} (M_{r_\rho}\varphi, \delta r_\rho)_{L^2(\Omega)} &= \sum_{s=1}^{S}\int_0^T \sum_{r=1}^{R} \varphi(x_r, z_r, t; x_s, z_s)\delta u(x, z, t; x_s, z_s)dt \\ &= \sum_{s=1}^{S}\int_0^T \int_\Omega \sum_{r=1}^{R} \varphi(x_r, z_r, t; x_s, z_s)\delta(x - x_s, z - z_s)\delta u(x, z, t; x_s, z_s)\ dx\ dz\ dt \\ &= \sum_{s=1}^{S}\int_0^T \int_\Omega \left(\frac{1}{\rho c^2}\frac{\partial^2 w_0}{\partial t^2} - \nabla(\frac{1}{\rho}\nabla w_0)\right)\delta u(x, z, t; x_s, z_s)\ dx\ dz\ dt \\ &= \sum_{s=1}^{S}\int_0^T \int_\Omega \left(\frac{1}{\rho c^2}\frac{\partial^2 \delta u}{\partial t^2} - \nabla(\frac{1}{\rho}\nabla \delta u)\right) w_0(x, z, t; x_s, z_s)\ dx\ dz\ dt \end{aligned}$$

$$= \sum_{s=1}^{S} \int_0^T \int_\Omega \left(\delta r_\rho \nabla(\frac{1}{\rho}\nabla u_0) - \nabla(\frac{\delta r_\rho}{\rho}\nabla u_0) \right) w_0(x,z,t;x_s,z_s)\ dx\ dz\ dt$$

$$= \int_\Omega \int_0^T \sum_{s=1}^{S} \left(w_0.\nabla(\frac{1}{\rho}\nabla u_0) + \frac{1}{\rho}\nabla w_0 \nabla u_0)dt \right) \delta r_\rho\ dx\ dz$$

whence

$$M_{r_\rho}\varphi = \sum_{s=1}^{S} \int_0^T w_0.\nabla(\frac{1}{\rho}\nabla u_0) + \frac{1}{\rho}\nabla w_0 \nabla u_0 dt$$

in the same way we have :

$$(M_{r_c}\varphi, \delta r_c)_{L^2(\Omega)} = \sum_{s=1}^{S}\sum_{r=1}^{R} (\varphi(x_r, z_r; x_s, z_s), D_{r_c}F(\rho, c, r_\rho, r_c).\delta r_c)_{L^2(0;T)}$$

$$= \sum_{s=1}^{S}\sum_{r=1}^{R} (\varphi(x_r, z_r; x_s, z_s), F(\rho, c, 0, r_c).\delta r_c)_{L^2(0;T)}$$

We have $F(\rho, c, 0, r_c) = \delta u$ solution of :

$$(11) \quad \begin{cases} \dfrac{1}{\rho c^2}\dfrac{\partial^2 \delta u}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla \delta u) = 2\delta r_c \nabla(\dfrac{1}{\rho}\nabla u_0) \\ \dfrac{1}{\rho c^2}\dfrac{\partial^2 u_0}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla u_0) = f(x, z, t; x_s, z_s) \\ \delta u(x, z, 0) = \dfrac{\partial \delta u}{\partial t}(x, z, 0) = 0 \\ \delta u(0, z, t) = \delta u(X, z, t) = \delta u(x, 0, t) = \delta u(x, Z, t) = 0 \\ u_0(x, z, 0) = \dfrac{\partial u_0}{\partial t}(x, z, 0) = 0 \\ u_0(0, z, t) = u_0(X, z, t) = u_0(x, 0, t) = u_0(x, Z, t) = 0 \end{cases}$$

then we have :

$$(M_{r_c}\varphi, \delta r_c)_{L^2(\Omega)} = \sum_{s=1}^{S} \int_0^T \sum_{r=1}^{R} \varphi(x_r, z_r, t; x_s, z_s)\delta u(x, z, t; x_s, z_s)dt$$

$$= \sum_{s=1}^{S} \int_0^T \int_\Omega \sum_{r=1}^{R} \varphi(x_r, z_r, t; x_s, z_s)\delta(x - x_s, z - z_s)\delta u(x, z, t; x_s, z_s) \; dx \, dz \, dt$$

$$= \sum_{s=1}^{S} \int_0^T \int_\Omega \left(\frac{1}{\rho c^2}\frac{\partial^2 w_0}{\partial t^2} - \nabla(\frac{1}{\rho}\nabla w_0)\right) \delta u(x, z, t; x_s, z_s) \; dx \, dz \, dt$$

$$= \sum_{s=1}^{S} \int_0^T \int_\Omega \left(\frac{1}{\rho c^2}\frac{\partial^2 \delta u}{\partial t^2} - \nabla(\frac{1}{\rho}\nabla \delta u)\right) w_0(x, z, t; x_s, z_s) \; dx \, dz \, dt$$

$$= \sum_{s=1}^{S} \int_0^T \int_\Omega \left(2\delta r_c \nabla(\frac{1}{\rho}\nabla u_0)\right) w_0(x, z, t; x_s, z_s) \; dx \, dz \, dt$$

$$= \int_\Omega \left(\int_0^T \sum_{s=1}^{S} 2w_0.\nabla(\frac{1}{\rho}\nabla u_0)dt\right) \delta r_c \; dx \, dz$$

whence

$$M_{r_c}\varphi = \sum_{s=1}^{S} \int_0^T 2w_0.\nabla(\frac{1}{\rho}\nabla u_0)dt$$

We then solve the normal equations by an iterative algorithm using Chebycheff polynomials (cf [1]).

3 Computation of the gradient

We assume now that the normal equations have been solved exactly, and therefore we have r_ρ and r_c as functions of ρ and c. Therefore

$$(12)\left\{\begin{aligned} \tilde{J}(\rho, c) &= J_{DS}(\rho, c, r_\rho(\rho, c), r_c(\rho, c)) \\ &= \frac{1}{2}\left\{||F(\rho, c, r_\rho(\rho, c), r_c(\rho, c)) - F_{data}||^2_{L^2(0;T)} + \sigma_\rho||\frac{\partial r_\rho(\rho, c)}{\partial x_s}||^2_{L^2(\Omega)}\right. \\ &+ \left.\sigma_c||\frac{\partial r_c(\rho, c)}{\partial x_s}||^2_{L^2(\Omega)} + \lambda_\rho^2||W r_\rho(\rho, c)||^2_{L^2(\Omega)} + \lambda_c^2||W r_c(\rho, c)||^2_{L^2(\Omega)}\right\} \end{aligned}\right.$$

The first derivative of $\tilde{J}$ due to a perturbation $(\delta\rho, \delta c)$ in (ρ, c) is given by :

$$D\tilde{J}(\rho, c)).(\delta\rho, \delta c) = D_\rho J_{DS}(\rho, c, r_\rho(\rho, c), r_c(\rho, c)).\delta\rho + D_c J_{DS}(\rho, c, r_\rho(\rho, c), r_c(\rho, c)).\delta c$$

$$+ \quad D_{r_\rho} J_{DS}(\rho, c, r_\rho(\rho, c), r_c(\rho, c)) \, (D_\rho r_\rho(\rho, c).\delta\rho + D_c r_\rho(\rho, c).\delta c)$$

$$+ \quad D_{r_c} J_{DS}(\rho, c, r_\rho(\rho, c), r_c(\rho, c)) \, (D_\rho r_c(\rho, c).\delta\rho + D_c r_c(\rho, c).\delta c)$$

But since we assumed that the normal equations have been solved exactly, cf (7) then :

$$\begin{cases} D_{r_\rho} J_{DS}(\rho, c, r_\rho(\rho, c), r_c(\rho, c)) & = & 0 \\ D_{r_c} J_{DS}(\rho, c, r_\rho(\rho, c), r_c(\rho, c)) & = & 0 \end{cases}$$

and we get the following simpler expression for the derivative of $\tilde{J}$:

$$D\tilde{J}(\rho, c)).(\delta\rho, \delta c) \;=\; D_\rho J_{DS}(\rho, c, r_\rho(\rho, c), r_c(\rho, c)).\delta\rho \,+\, D_c J_{DS}(\rho, c, r_\rho(\rho, c), r_c(\rho, c)).\delta c$$

Since

$$\tilde{J}(\rho, c) \;=\; \frac{1}{2} ||F(\rho, c, r_\rho, r_c) - F_{data}||^2_{L^2(0;T)} \,+\, \xi(r_\rho, r_c)$$

where ξ does not depend explicitly on ρ and c, we have :

$$\begin{cases} D_\rho J_{DS}(\rho, c, r_\rho, r_c).\delta\rho & = & (D_\rho F(\rho, c, r_\rho, r_c).\delta\rho, F(\rho, c, r_\rho, r_c) - F_{data})_{L^2(0;T)} \\ D_c J_{DS}(\rho, c, r_\rho, r_c).\delta c & = & (D_c F(\rho, c, r_\rho, r_c).\delta c, F(\rho, c, r_\rho, r_c) - F_{data})_{L^2(0;T)} \end{cases}$$

Now to compute the gradients of $\tilde{J}$ with respect to ρ and c, we must find two bilinear forms B_ρ and B_c such that

$$(D_\rho F(\rho, c, r_\rho, r_c).\delta\rho, F(\rho, c, r_\rho, r_c) - F_{data})_{L^2(0;T)} \;=\; (\delta\rho, B_\rho.(F(\rho, c, r_\rho, r_c) - F_{data}))_{L^2(\Omega)}$$

$$(D_c F(\rho, c, r_\rho, r_c).\delta c, F(\rho, c, r_\rho, r_c) - F_{data})_{L^2(0;T)} \;=\; (\delta c, B_c(F(\rho, c, r_\rho, r_c) - F_{data}))_{L^2(\Omega)}$$

Remark

r_ρ and r_c being chosen as the solution of the normal equations, they are fixed. To enhance this fact, we use a different notation and from now on we will use q_ρ for r_ρ and q_c for r_c.

We know that $u = F(\rho, c, q_p, q_c)$ is the solution of :

$$
(13)\quad
\begin{cases}
\dfrac{1}{\rho c^2}\dfrac{\partial^2 u}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla u) = (q_p + 2q_c)\,\nabla(\dfrac{1}{\rho}\nabla u_0) - \nabla(\dfrac{q_\rho}{\rho}\nabla u_0) \\[2ex]
\dfrac{1}{\rho c^2}\dfrac{\partial^2 u_0}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla u_0) = f(x, z, t; x_s, z_s) \\[2ex]
u(x,z,0) = u_0(x,z,0) = \dfrac{\partial u}{\partial t}(x,z,0) = \dfrac{\partial u_0}{\partial t}(x,z,0) = 0 \\[2ex]
u_0(0,z,t) = u_0(x,0,t) = u_0(X,z,t) = u_0(x,Z,t) = 0 \\[2ex]
u(0,z,t) = u(x,0,t) = u(X,z,t) = u(x,Z,t) = 0
\end{cases}
$$

With a little algebra it is easy to see that $\delta u = D_\rho F(\rho, c, q_p, q_c).\delta\rho$ is the solution of

$$
(14)\begin{cases}
\dfrac{1}{\rho c^2}\dfrac{\partial^2 \delta u}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla\delta u) = -\nabla(\dfrac{\delta\rho}{\rho^2}\nabla u) \\[1ex]
\qquad + \dfrac{\delta\rho}{\rho}\left\{\nabla(\dfrac{1}{\rho}\nabla u) + (q_p + 2q_c)\,\nabla(\dfrac{1}{\rho}\nabla u_0) - \nabla(\dfrac{q_p}{\rho}\nabla u_0)\right\} \\[1ex]
\qquad - (q_p + 2q_c)\,\nabla(\dfrac{\delta\rho}{\rho^2}\nabla u_0) + \nabla(\dfrac{q_p}{\rho}\dfrac{\delta\rho}{\rho}\nabla u_0) \\[1ex]
\qquad + (q_p + 2q_c)\,\nabla(\dfrac{1}{\rho}\nabla\delta u_0) - \nabla(\dfrac{q_p}{\rho}\nabla\delta u_0) \\[2ex]
\dfrac{1}{\rho c^2}\dfrac{\partial^2 \delta u_0}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla\delta u_0) = \dfrac{\delta\rho}{\rho}\nabla(\dfrac{1}{\rho}\nabla u_0) - \nabla(\dfrac{\delta\rho}{\rho^2}\nabla u_0) \\[2ex]
\delta u(x,z,0) = \delta u_0(x,z,0) = \dfrac{\partial\delta u}{\partial t}(x,z,0) = \dfrac{\partial\delta u_0}{\partial t}(x,z,0) = 0 \\[2ex]
\delta u_0(0,z,t) = \delta u_0(x,0,t) = \delta u_0(X,z,t) = \delta u_0(x,Z,t) = 0 \\[2ex]
\delta u(0,z,t) = \delta u(x,0,t) = \delta u(X,z,t) = \delta u(x,Z,t) = 0
\end{cases}
$$

Given a seismogram φ, we can now evaluate the following quantity :

$$(D_\rho F(\rho,c,r_\rho,r_c).\delta\rho,\varphi)_{L^2(0;T)} = \sum_{s=1}^{S}\sum_{r=1}^{R}(\varphi(x_r,z_r;x_s,z_s),\delta u(x_r,z_r;x_s,z_s))_{L^2(0;T)}$$

$$= \sum_{s=1}^{S}\int_0^T\int_\Omega\sum_{r=1}^{R}\varphi(x_r,z_r,t;x_s,z_s)\ \delta u(x,z,t;x_s,z_s)\ \delta(x-x_s,z-z_s)\ dx\ dz\ dt$$

$$= \sum_{s=1}^{S}\int_0^T\int_\Omega\left(\frac{1}{\rho c^2}\frac{\partial^2 w_0}{\partial t^2}-\nabla(\frac{1}{\rho}\nabla w_0)\right)\delta u(x,z,t;x_s,z_s)\ dx\ dz\ dt$$

$$= \sum_{s=1}^{S}\int_0^T\int_\Omega w_0(x,z,t;x_s,z_s)\left(\frac{1}{\rho c^2}\frac{\partial^2\delta u}{\partial t^2}-\nabla(\frac{1}{\rho}\nabla\delta u)\right)\ dx\ dz\ dt$$

$$= \sum_{s=1}^{S}\int_0^T\int_\Omega\frac{w_0}{\rho}\left(\nabla(\frac{1}{\rho}\nabla u)+(q_\rho+2q_c)\nabla(\frac{1}{\rho}\nabla u_0)-\nabla(\frac{q_\rho}{\rho}\nabla u_0)\right)\delta\rho\ dx\ dz\ dt$$

$$+\int_0^T\int_\Omega\frac{1}{\rho^2}(\nabla w_0\nabla u+\nabla((q_\rho+2q_c)w_0)\nabla u_0-q_\rho\nabla w_0\nabla u_0)\,\delta\rho\ dx\ dz\ dt$$

$$+\int_0^T\int_\Omega\left(\nabla(\frac{1}{\rho}\nabla(q_\rho+2q_c)w_0)-\nabla(\frac{q_\rho}{\rho}\nabla w_0)\right)\delta u_0\ dx\ dz\ dt$$

where w_0 is the solution of (10). To find the expression of the gradient of $\tilde{J}$ with respect to ρ, we need to work on the last integral. We introduce another adjoint state w defined by :

$$(15)\qquad \begin{cases} \dfrac{1}{\rho c^2}\dfrac{\partial^2 w}{\partial t^2}-\nabla(\dfrac{1}{\rho}\nabla w) = \nabla(\dfrac{1}{\rho}\nabla(q_\rho+2q_c)w_0)-\nabla(\dfrac{q_\rho}{\rho}\nabla w_0) \\ w(x,z,T)=\dfrac{\partial w}{\partial t}(x,z,T)=0 \\ w(0,z,t)=w(X,z,t)=w(x,0,t)=w(x,Z,t)=0 \end{cases}$$

We can then pursue the previous calculation :

$$
\begin{aligned}
I &= \sum_{s=1}^{S} \int_0^T \int_\Omega \left(\nabla(\frac{1}{\rho}\nabla(q_p + 2q_c)w_0) - \nabla(\frac{q_p}{\rho}\nabla w_0) \right) \delta u_0 \;\; dx\; dz\; dt \\
&= \sum_{s=1}^{S} \int_0^T \int_\Omega \left(\frac{1}{\rho c^2}\frac{\partial^2 w}{\partial t^2} - \nabla(\frac{1}{\rho}\nabla w) \right) \delta u_0 \;\; dx\; dz\; dt \\
&= \sum_{s=1}^{S} \int_0^T \int_\Omega w \left(\frac{1}{\rho c^2}\frac{\partial^2 \delta u_0}{\partial t^2} - \nabla(\frac{1}{\rho}\nabla \delta u_0) \right) \;\; dx\; dz\; dt \\
&= \sum_{s=1}^{S} \int_\Omega \left(\int_0^T \frac{w}{\rho}\nabla(\frac{1}{\rho}\nabla u_0) + \frac{1}{\rho^2}\nabla w \nabla u_0 dt \right) \delta\rho \;\; dx\; dz
\end{aligned}
$$

whence :

$$
(16) \quad \left\{
\begin{aligned}
\nabla_\rho \tilde{J} \;&=\; \sum_{s=1}^{S} \int_0^T \frac{w_0}{\rho} \left(\nabla(\frac{1}{\rho}\nabla u) + (q_p + 2q_c)\,\nabla(\frac{1}{\rho}\nabla u_0) - \nabla(\frac{q_p}{\rho}\nabla u_0) \right) \; dt \\
&+\; \sum_{s=1}^{S} \int_0^T \frac{1}{\rho^2} (\nabla w_0 \nabla u + \nabla((q_p + 2q_c)w_0)\nabla u_0 - q_p \nabla w_0 \nabla u_0) \; dt \\
&+\; \sum_{s=1}^{S} \int_0^T \left(\frac{w}{\rho}\nabla(\frac{1}{\rho}\nabla u_0) + \frac{1}{\rho^2}\nabla w \nabla u_0 \right) \; dt
\end{aligned}
\right.
$$

From (13) we can derive the equation verified by $\delta u = D_c F(\rho, c, q_\rho, q_c).\delta c$:

$$(17)\quad \begin{cases} \dfrac{1}{\rho c^2}\dfrac{\partial^2 \delta u}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla \delta u) = 2\dfrac{\delta c}{c}\nabla(\dfrac{1}{\rho}\nabla u) \\ \qquad +2\dfrac{\delta c}{c}\left\{(q_\rho + 2q_c)\nabla(\dfrac{1}{\rho}\nabla u_0) - \nabla(\dfrac{q_\rho}{\rho}\nabla u_0)\right\} \\ \qquad +(q_\rho + 2q_c)\nabla(\dfrac{1}{\rho}\nabla \delta u_0) - \nabla(\dfrac{q_\rho}{\rho}\nabla \delta u_0) \\ \dfrac{1}{\rho c^2}\dfrac{\partial^2 \delta u_0}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla \delta u_0) = 2\dfrac{\delta c}{c}\nabla(\dfrac{1}{\rho}\nabla u_0) \\ \delta u(x,z,0) = \delta u_0(x,z,0) = \dfrac{\partial \delta u}{\partial t}(x,z,0) = \dfrac{\partial \delta u_0}{\partial t}(x,z,0) = 0 \\ \delta u_0(0,z,t) = \delta u_0(x,0,t) = \delta u_0(X,z,t) = \delta u_0(x,Z,t) = 0 \\ \delta u(0,z,t) = \delta u(x,0,t) = \delta u(X,z,t) = \delta u(x,Z,t) = 0 \end{cases}$$

Given a seismogram φ, we can now evaluate the following quantity :

$$(D_c F(\rho, c, r_\rho, r_c).\delta c, \varphi)_{L^2(0;T)} = \sum_{s=1}^{S}\sum_{r=1}^{R}(\varphi(x_r, z_r; x_s, z_s), \delta u(x_r, z_r; x_s, z_s))_{L^2(0;T)}$$

$$= \sum_{s=1}^{S}\int_0^T\int_\Omega \sum_{r=1}^{R}\varphi(x_r, z_r, t; x_s, z_s)\delta u(x, z, t; x_s, z_s)\delta(x - x_s, z - z_s)\ dx\ dz\ dt$$

$$= \sum_{s=1}^{S}\int_0^T\int_\Omega \left(\frac{1}{\rho c^2}\frac{\partial^2 w_0}{\partial t^2} - \nabla(\frac{1}{\rho}\nabla w_0)\right)\delta u(x, z, t; x_s, z_s)\ dx\ dz\ dt$$

$$= \sum_{s=1}^{S}\int_0^T\int_\Omega w_0(x, z, t; x_s, z_s)\left(\frac{1}{\rho c^2}\frac{\partial^2 \delta u}{\partial t^2} - \nabla(\frac{1}{\rho}\nabla \delta u)\right)\ dx\ dz\ dt$$

$$= \sum_{s=1}^{S} \int_0^T \int_\Omega 2\frac{w_0}{c}\left(\nabla(\frac{1}{\rho}\nabla u) + (q_\rho + 2q_c)\,\nabla(\frac{1}{\rho}\nabla u_0) - \nabla(\frac{q_\rho}{\rho}\nabla u_0)\right)\delta c \;\; dx\, dz\, dt$$

$$+ \int_0^T \int_\Omega w_0\,(q_\rho + 2q_c)\,\nabla w_0 \nabla(\frac{1}{\rho}\nabla\delta u_0 - \nabla(\frac{q_\rho}{\rho}\nabla\delta u_0) \;\; dx\, dz\, dt$$

$$= \sum_{s=1}^{S} \int_0^T \int_\Omega 2\frac{w_0}{c}\left(\nabla(\frac{1}{\rho}\nabla u) + (q_\rho + 2q_c)\,\nabla(\frac{1}{\rho}\nabla u_0) - \nabla(\frac{q_\rho}{\rho}\nabla u_0)\right)\delta c \;\; dx\, dz\, dt$$

$$+ \int_0^T \int_\Omega \delta u_0 \left(\nabla(\frac{1}{\rho}\nabla(q_\rho + 2q_c)w_0) - \nabla(\frac{q_\rho}{\rho}\nabla w_0)\right) \;\; dx\, dz\, dt$$

where w_0 is the solution of (10). Introducing w the solution of (15), we can write the second integral as :

$$\sum_{s=1}^{S} \int_0^T \int_\Omega \left(\nabla(\frac{1}{\rho}\nabla(q_\rho + 2q_c)w_0 - \nabla(\frac{q_\rho}{\rho}\nabla w_0)\right)\delta u_0 \;\; dx\, dz\, dt$$

$$= \sum_{s=1}^{S} \int_0^T \int_\Omega \left(\frac{1}{\rho c^2}\frac{\partial^2 w}{\partial t^2} - \nabla(\frac{1}{\rho}\nabla w)\right)\delta u_0 \;\; dx\, dz\, dt$$

$$= \sum_{s=1}^{S} \int_0^T \int_\Omega w\left(\frac{1}{\rho c^2}\frac{\partial^2 \delta u_0}{\partial t^2} - \nabla(\frac{1}{\rho}\nabla\delta u_0)\right) \;\; dx\, dz\, dt$$

$$= \sum_{s=1}^{S} \int_\Omega \left(\int_0^T 2\frac{w}{c}\nabla(\frac{1}{\rho}\nabla u_0) \;\; dt\right)\delta c \;\; dx\, dz$$

whence :

$$(18)\begin{cases} \nabla_c \bar{J} &= \displaystyle\sum_{s=1}^{S} \int_0^T 2\frac{w_0}{c}\left(\nabla(\frac{1}{\rho}\nabla u) + (q_\rho + 2q_c)\nabla(\frac{1}{\rho}\nabla u_0) - 2\frac{w_0}{c}\nabla(\frac{q_\rho}{\rho}\nabla u_0)\right) \;\; dt \\ & + \displaystyle\sum_{s=1}^{S} \int_0^T 2\frac{w}{c}\nabla(\frac{1}{\rho}\nabla u_0) \;\; dt \end{cases}$$

4 Implementation of the Gradient

We now turn to the implementation aspects of the computation of the two gradients obtained above. We are going to use a finite difference method to compute the different wave fields we need. We summarize below the expressions of the two gradients and the equations we need to discretize to compute them.

$$\begin{aligned}
\nabla_\rho \bar{J} &= \sum_{s=1}^{S} \int_0^T \frac{w_0}{\rho} \left(\nabla(\frac{1}{\rho}\nabla u) + (q_\rho + 2q_c)\nabla(\frac{1}{\rho}\nabla u_0) - \nabla(\frac{q_\rho}{\rho}\nabla u_0) \right) dt \\
&+ \sum_{s=1}^{S} \int_0^T \frac{1}{\rho^2} (\nabla w_0 \nabla u + \nabla((q_\rho + 2q_c) w_0) \nabla u_0 - q_\rho \nabla w_0 \nabla u_0) \, dt \\
&+ \sum_{s=1}^{S} \int_0^T \left(\frac{w}{\rho} \nabla(\frac{1}{\rho}\nabla u_0) + \frac{1}{\rho^2} \nabla w \nabla u_0 \right) dt \\
\nabla_c \bar{J} &= \sum_{s=1}^{S} \int_0^T 2\frac{w_0}{c} \left(\nabla(\frac{1}{\rho}\nabla u) + (q_\rho + 2q_c)\nabla(\frac{1}{\rho}\nabla u_0) - 2\frac{w_0}{c}\nabla(\frac{q_\rho}{\rho}\nabla u_0) \right) dt \\
&+ \sum_{s=1}^{S} \int_0^T 2\frac{w}{c} \nabla(\frac{1}{\rho}\nabla u_0) \, dt
\end{aligned}$$

where the two direct states u_0, u are solutions of :

$$\left\{ \begin{aligned}
&\frac{1}{\rho c^2}\frac{\partial^2 u_0}{\partial t^2} - \nabla(\frac{1}{\rho}\nabla u_0) = f(x, z, t; x_s, z_s) \\
&u_0(x, z, 0) = \frac{\partial u_0}{\partial t}(x, z, 0) = 0 \\
&u_0(0, z, t) = u_0(X, z, t) = u_0(x, 0, t) = u_0(x, Z, t) = 0
\end{aligned} \right.$$

$$\left\{ \begin{aligned}
&\frac{1}{\rho c^2}\frac{\partial^2 u}{\partial t^2} - \nabla(\frac{1}{\rho}\nabla u) = (q_\rho + 2q_c)\nabla(\frac{1}{\rho}\nabla u_0) - \nabla(\frac{q_\rho}{\rho}\nabla u_0) \\
&u(x, z, 0) = \frac{\partial u}{\partial t}(x, z, 0) = 0 \\
&u(0, z, t) = u(X, z, t) = u(x, 0, t) = u(x, Z, t) = 0
\end{aligned} \right.$$

and the two adjoint states w_0 and w are solutions of :

$$\begin{cases} \dfrac{1}{\rho c^2}\dfrac{\partial^2 w_0}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla w_0) = \displaystyle\sum_{r=1}^{R} (F(\rho, c, q_\rho, q_c)(x_r, z_r, t; x_s, z_s) - F_{data}(t))\,\delta(x - x_s, z - z_s) \\ w_0(x, z, T) = \dfrac{\partial w_0}{\partial t}(x, z, T) = 0 \\ w_0(0, z, t) = w_0(X, z, t) = w_0(x, 0, t) = w_0(x, Z, t) = 0 \end{cases}$$

$$\begin{cases} \dfrac{1}{\rho c^2}\dfrac{\partial^2 w}{\partial t^2} - \nabla(\dfrac{1}{\rho}\nabla w) = \nabla(\dfrac{1}{\rho}\nabla(q_\rho + 2q_c)w_0) - \nabla(\dfrac{q_\rho}{\rho}\nabla w_0) \\ w(x, z, T) = \dfrac{\partial w}{\partial t}(x, z, T) = 0 \\ w(0, z, t) = w(X, z, t) = w(x, 0, t) = w(x, Z, t) = 0 \end{cases}$$

The finite difference schemes used to simulate those wave fields are of order 2 in time and 2L in space, where L is the number of points used in the calculation of the derivative. Those schemes are described in detail in [4].
In the previous calculations, we repeatedly used the vanishing of u_0, u, w_0 and w on the boundary of the domain. In order for these calculations to carry over to the discretized equations we need to make sure that $u_{0,h}$, u_h, $w_{0,h}$ and w_h, discrete equivalents of u_0, u, w_0 and w, have the same property, with the same consequences.
For this we use the well known 'image principle', by extending $u_{0,h}$, u_h, $w_{0,h}$ and w_h outside the domain.
First, we define some functional spaces, to which $u_{0,h}$, u_h, $w_{0,h}$ and w_h will belong.

$$L^2_{o,o} = \left\{ \varphi \in L^2(\Omega) \ / \ \varphi = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} \varphi_{i,j}\ 1_{[(i-\frac{1}{2})\Delta x,(i+\frac{1}{2})\Delta x]\times[(j-\frac{1}{2})\Delta z,(j+\frac{1}{2})\Delta z]}(x, z) \right\}$$

$$L^2_{*,*} = \left\{ \varphi \in L^2(\Omega) \ / \ \varphi = \sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1} \varphi_{i+\frac{1}{2},j+\frac{1}{2}}\ 1_{[i\Delta x,(i+1)\Delta x]\times[j\Delta z,(j+1)\Delta z]}(x, z) \right\}$$

$$L^2_{o,*} = \left\{ \varphi \in L^2(\Omega) \ / \ \varphi = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y-1} \varphi_{i,j+\frac{1}{2}} \ 1_{[(i-\frac{1}{2})\Delta x,(i+\frac{1}{2})\Delta x]\times[j\Delta z,(j+1)\Delta z]}(x,z) \right\}$$

$$L^2_{*,o} = \left\{ \varphi \in L^2(\Omega) \ / \ \varphi = \sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y} \varphi_{i+\frac{1}{2},j} \ 1_{[i\Delta x,(i+1)\Delta x]\times[(j-\frac{1}{2})\Delta z,(j+\frac{1}{2})\Delta z]}(x,z) \right\}$$

where

$$1_{[a,b]\times[c,d]}(x,z) = \begin{cases} 1 & (x,z) \in [a,b]\times[c,d] \\ 0 & (x,z) \notin [a,b]\times[c,d] \end{cases}$$

We approximate the first derivative by the following operator :

$$\begin{aligned} A^o_x \ : \ L^2_{o,o} &\longrightarrow L^2_{*,o} \\ u &\longmapsto A^o_x u(i+\tfrac{1}{2},j) = \sum_{l=1}^{L} \frac{\beta_l}{\Delta x}[u(i+l,j) - u(i-l+1,j)] \end{aligned}$$

A^o_x is a finite difference approximation of order 2L in $((i+\frac{1}{2})\Delta x, j\Delta z)$ of the quantity $\frac{\partial u}{\partial x}$, with the coefficients $(\beta_l)_{l=1..L}$ defined in appendix 1. The exponent refers to the departure set $L^2_{o,o}$; the subscript to the direction of differentiation. Similarly we define :

$$\begin{aligned} A^o_z \ : \ L^2_{o,o} &\longrightarrow L^2_{o,*} \\ u &\longmapsto A^o_z u(i,j+\tfrac{1}{2}) = \sum_{l=1}^{L} \frac{\beta_l}{\Delta z}[u(i,j+l) - u(i,j-l+1)] \end{aligned}$$

$$\begin{aligned} A^*_x \ : \ L^2_{*,*} &\longrightarrow L^2_{o,*} \\ v &\longmapsto A^*_x v(i,j+\tfrac{1}{2}) = \sum_{l=1}^{L} \frac{\beta_l}{\Delta x}[v(i+l+\frac{1}{2},j+\frac{1}{2}) - v(i-l+\frac{1}{2},j+\frac{1}{2})] \end{aligned}$$

$$\begin{aligned} A^*_z \ : \ L^2_{*,*} &\longrightarrow L^2_{*,o} \\ v &\longmapsto A^*_z v(i,j+\tfrac{1}{2}) = \sum_{l=1}^{L} \frac{\beta_l}{\Delta z}[u(i+\frac{1}{2},j+l+\frac{1}{2}) - u(i+\frac{1}{2},j-l+\frac{1}{2})] \end{aligned}$$

We approximate the quantity $\nabla(\frac{1}{\rho}\nabla u)$ by $\nabla_h(\frac{1}{\rho}\nabla_h u) = -{}^t A^*_x(\frac{1}{\rho}A^o_x u) -{}^t A^*_z(\frac{1}{\rho}A^o_z u)$

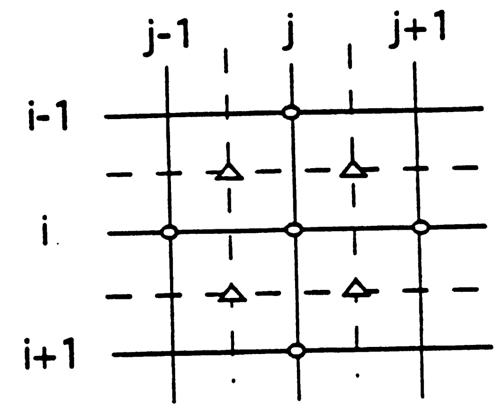

Fig 1 : The original and shifted grids

The finite difference operators use $2L$ points to compute the derivative. For instance, $A_x^o u$ approximates $\frac{\partial u}{\partial x}$ in $((i+\frac{1}{2})\Delta x, j\Delta z)$ with L points to the right, and L points to the left of $((i+\frac{1}{2})\Delta x, j\Delta z)$.
Therefore when we reach the boundary of the domain, for instance, we miss points to compute our derivative. A way to solve that problem is to extend the function we need outside the domain Ω. There are many ways to do so, but we keep in mind that we want our discretized functions to be subject to Dirichlet boundary conditions.
We are going to derive the way to extend the discretized functions, so that they will vanish on the boundary, and the operators $\nabla_h(\frac{1}{\rho}\nabla_h u)$ will be self adjoint.
Since the derivatives are taken, each time in one direction, it is equivalent to consider a unidimensionnal problem. We consider $\Omega = [0, X]$ and we set $\Omega_o = [1, 2..N]$ and $\Omega_* = [1, 2..N-1]$. We define the finite difference operators as follows :

$$\begin{array}{rcll} A^o & : & L^2(\Omega_o) & \longrightarrow L^2(\Omega_*) \\ & & u & \longmapsto A^o u(i+\frac{1}{2}) = \sum_{l=1}^{L} a_l[u_{i+l} - u_{i-l+1}] \end{array}$$

$$\begin{array}{rcll} A^* & : & L^2(\Omega_*) & \longrightarrow L^2(\Omega_o) \\ & & v & \longmapsto A^* v(i) = \sum_{l=1}^{L} a_l[u_{i+l-\frac{1}{2}} - u_{i-l+\frac{1}{2}}] \end{array}$$

with $a_l = \beta_l/\Delta x$ and :

$$
\begin{aligned}
L^2(\Omega_o) &= \left\{ \varphi \in L^2(\Omega) \ / \ \varphi = \sum_{i=1}^{N_x} \varphi_i \ 1_{[(i-\frac{1}{2})\Delta x,(i+\frac{1}{2})\Delta x]}(x) \right\} \\
L^2(\Omega_*) &= \left\{ \varphi \in L^2(\Omega) \ / \ \varphi = \sum_{i=1}^{N_x-1} \varphi_{i+\frac{1}{2}} \ 1_{[i\Delta x,(i+1)\Delta x]} \right\}
\end{aligned}
$$

Thoses spaces are provided with the usual scalar products defined by :

$$
\begin{aligned}
(f,g)_{L^2(\Omega_o)} &= (f,g)_o = \sum_{i=1}^{N} f_i g_i \Delta x \Delta z \\
(f,g)_{L^2(\Omega_*)} &= (f,g)_* = \sum_{i=1}^{N-1} f_{i+\frac{1}{2}} g_{i+\frac{1}{2}} \Delta x \Delta z
\end{aligned}
$$

We suppose that the boundaries of the domain are located in $i = 1$ and $i = N$, so that $u_1 = u_N = 0$. We extend $u \in \Omega_o$, outside Ω_o by the following procedure :

$$
\begin{aligned}
u_{1-k} &= -u_{1+k} \qquad k = 0..L-1 \\
u_{N+k} &= -u_{N-k} \qquad k = 0..L-1
\end{aligned}
$$

That is we skew-symetrize u, at the boundary. Now we want to find under what conditions we have

$$
(Au, v)_* = (v, {}^tAu)_o
$$

That is an integration by part without boundary terms. We have :

$$\begin{aligned}
(Au,v)_* &= \sum_{i=1}^{N-1} Au_{i+\frac{1}{2}}\, v_{i+\frac{1}{2}} \\
&= \sum_{i=1}^{N-1}\sum_{l=1}^{L} a_l(u_{i+l}-u_{i-l+1})\, v_{i+\frac{1}{2}} \\
&= \sum_{l=1}^{L} a_l\left(\sum_{i=1}^{N-1} u_{i+l}\, v_{i+\frac{1}{2}} - \sum_{i=1}^{N-1} u_{i-l+1}\, v_{i+\frac{1}{2}}\right) \\
&= \sum_{l=1}^{L} a_l\left(\sum_{j=l+1}^{N+l-1} u_j\, v_{j-l+\frac{1}{2}} - \sum_{j=1-l}^{N-l} u_j\, v_{j+l-\frac{1}{2}}\right) \\
&= \sum_{i=1}^{N}\sum_{l=1}^{L} a_l(v_{j-l+\frac{1}{2}} - v_{j+l-\frac{1}{2}})\, u_j - \sum_{l=1}^{L} a_l\left(\sum_{j=1}^{l} u_j\, v_{j-l+\frac{1}{2}} + \sum_{j=2-l}^{0} u_j\, v_{j+l-\frac{1}{2}}\right) \\
&+ \sum_{l=1}^{L} a_l\left(\sum_{j=N+1}^{N+l-1} u_j\, v_{j-l+\frac{1}{2}} + \sum_{j=N-l+1}^{N} u_j\, v_{j+l-\frac{1}{2}}\right) \\
&= (v,\, {}^tAu)_o - \sum_{l=1}^{L} a_l\left(\sum_{j=1}^{l} u_j\, v_{j-l+\frac{1}{2}} + \sum_{j=2-l}^{0} u_j\, v_{j+l-\frac{1}{2}}\right) \\
&+ \sum_{l=1}^{L} a_l\left(\sum_{j=N+1}^{N+l-1} u_j\, v_{j-l+\frac{1}{2}} + \sum_{j=N-l+1}^{N} u_j\, v_{j+l-\frac{1}{2}}\right)
\end{aligned}$$

Therefore we have boundary terms,given by

$$\begin{aligned}
B_1 &= \sum_{l=1}^{L} a_l\left(\sum_{j=1}^{l} u_j\, v_{j-l+\frac{1}{2}} + \sum_{j=2-l}^{0} u_j\, v_{j+l-\frac{1}{2}}\right) \\
B_2 &= \sum_{l=1}^{L} a_l\left(\sum_{j=N+1}^{N+l-1} u_j\, v_{j-l+\frac{1}{2}} + \sum_{j=N-l+1}^{N} u_j\, v_{j+l-\frac{1}{2}}\right)
\end{aligned}$$

We have

$$
\begin{aligned}
B_1 &= \sum_{l=1}^{L} a_l \left(\sum_{j=1}^{l} u_j \, v_{j-l+\frac{1}{2}} + \sum_{j=2-l}^{0} u_j \, v_{j+l-\frac{1}{2}} \right) \\
&= \sum_{l=1}^{L} a_l \left(\sum_{j=2}^{l} u_j \, v_{j-l+\frac{1}{2}} + \sum_{k=1}^{l-1} u_{1-k} \, v_{-k+l+\frac{1}{2}} \right) \\
&= \sum_{l=1}^{L} a_l \left(\sum_{k=1}^{l-1} u_{1+k} \, v_{j-l+\frac{1}{2}} + u_{1-k} \, v_{-k+l+\frac{1}{2}} \right) \\
&= \sum_{l=1}^{L} a_l \left(\sum_{k=1}^{l-1} u_{1+k} \, (v_{j-l+\frac{1}{2}} - v_{-k+l+\frac{1}{2}}) \right)
\end{aligned}
$$

and

$$
\begin{aligned}
B_2 &= \sum_{l=1}^{L} a_l \left(\sum_{j=N+1}^{N+l-1} u_j \, v_{j-l+\frac{1}{2}} + \sum_{j=N-l+1}^{N} u_j \, v_{j+l-\frac{1}{2}} \right) \\
&= \sum_{l=1}^{L} a_l \left(\sum_{k=1}^{l-1} u_{N+k} \, v_{N+k-l+\frac{1}{2}} + \sum_{k=0}^{l-1} u_{N-k} \, v_{N-k+l-\frac{1}{2}} \right) \\
&= \sum_{l=1}^{L} a_l \left(\sum_{k=1}^{l-1} u_{N+k} \, v_{N+k-l+\frac{1}{2}} + \sum_{k=1}^{l-1} u_{N-k} \, v_{N-k+l-\frac{1}{2}} \right) \\
&= \sum_{l=1}^{L} a_l \left(\sum_{k=1}^{l-1} u_{N-k} (v_{N-k+l-\frac{1}{2}} - v_{N+k-l+\frac{1}{2}}) \right)
\end{aligned}
$$

We see that if we extend v outside Ω_* by symetry, that is :

$$
\begin{aligned}
v_{-k+\frac{1}{2}} &= v_{k+\frac{1}{2}} && k = 0..L-1 \\
v_{N+\frac{1}{2}+k} &= v_{N+\frac{1}{2}-k} && k = 0..L-1
\end{aligned}
$$

we annihilate the boundary terms B_1 and B_2.

To summarize, with :

$$
\begin{aligned}
u_{1-k} &= -u_{1+k} & k = 0..L-1 \\
u_{N+k} &= -u_{N-k} & k = 0..L-1 \\
v_{-k+\frac{1}{2}} &= v_{k+\frac{1}{2}} & k = 0..L-1 \\
v_{N+\frac{1}{2}+k} &= v_{N+\frac{1}{2}-k} & k = 0..L-1
\end{aligned}
$$

the implementation of $\nabla(\frac{1}{\rho}\nabla u)$ by $\nabla_h(\frac{1}{\rho}\nabla_h u) = -{}^t A_x^*(\frac{1}{\rho}A_x^o u) -{}^t A_z^*(\frac{1}{\rho}A_z^o u)$ defines an self-adjoint operator, and the integrations by parts carried out in the previous section for the computation of the gradients, carriy over to the discrete case.
Therefore we define $u_{0,h}$, u_h as solutions of

$$
\left\{
\begin{aligned}
&\frac{1}{\rho c^2}\frac{\partial^2 u_{0,h}}{\partial t^2} - \nabla_h(\frac{1}{\rho}\nabla_h u_{0,h}) = f(x,z,t;x_s,z_s) \\
&u_{0,h}(x,z,0) = \frac{\partial u_{0,h}}{\partial t}(x,z,0) = 0 \\
&u_{0,h}(0,z,t) = u_{0,h}(X,z,t) = u_{0,h}(x,0,t) = u_{0,h}(x,Z,t) = 0
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
&\frac{1}{\rho c^2}\frac{\partial^2 u_h}{\partial t^2} - \nabla_h(\frac{1}{\rho}\nabla_h u_h) = (q_\rho + 2q_c)\nabla_h(\frac{1}{\rho}\nabla_h u_{0,h}) - \nabla_h(\frac{q_\rho}{\rho}\nabla_h u_{0,h}) \\
&u_h(x,z,0) = \frac{\partial u_h}{\partial t}(x,z,0) = 0 \\
&u_h(0,z,t) = u_h(X,z,t) = u_h(x,0,t) = u_h(x,Z,t) = 0
\end{aligned}
\right.
$$

and the two adjoint states $w_{0,h}$ and w_h as solutions of :

$$\begin{cases} \dfrac{1}{\rho c^2}\dfrac{\partial^2 w_{0,h}}{\partial t^2} - \nabla_h(\dfrac{1}{\rho}\nabla_h w_{0,h}) = \displaystyle\sum_{r=1}^{R}(F(\rho, c, q_p, q_c)(x_r, z_r, t; x_s, z_s) - F_{data}(t))\,\delta(x - x_s, z - z_s) \\ w_{0,h}(x,z,T) = \dfrac{\partial w_{0,h}}{\partial t}(x,z,T) = 0 \\ w_{0,h}(0,z,t) = w_{0,h}(X,z,t) = w_{0,h}(x,0,t) = w_{0,h}(x,Z,t) = 0 \end{cases}$$

$$\begin{cases} \dfrac{1}{\rho c^2}\dfrac{\partial^2 w_h}{\partial t^2} - \nabla_h(\dfrac{1}{\rho}\nabla_h w_h) = \nabla_h(\dfrac{1}{\rho}\nabla_h(q_p + 2q_c)w_{0,h}) - \nabla_h(\dfrac{q_p}{\rho}\nabla_h w_{0,h}) \\ w_h(x,z,T) = \dfrac{\partial w_h}{\partial t}(x,z,T) = 0 \\ w_h(0,z,t) = w_h(X,z,t) = w_h(x,0,t) = w_h(x,Z,t) = 0 \end{cases}$$

The two gradients are now given by

$$\begin{aligned} \nabla_{\rho,h}\tilde{J} &= \sum_{s=1}^{S}\sum_{n=1}^{N}\frac{w_{0,h}}{\rho}\left(\nabla_h(\frac{1}{\rho}\nabla_h u_h) + (q_p + 2q_c)\nabla_h(\frac{1}{\rho}\nabla_h u_{0,h}) - \nabla_h(\frac{q_p}{\rho}\nabla_h u_{0,h})\right)\,\Delta t \\ &+ \sum_{s=1}^{S}\sum_{n=1}^{N}\frac{1}{\rho^2}(\nabla_h w_{0,h}\nabla_h u_h + \nabla_h((q_p + 2q_c)w_{0,h})\nabla_h u_{0,h} - q_p\nabla_h w_{0,h}\nabla_h u_{0,h})\,\Delta t \\ &+ \sum_{s=1}^{S}\sum_{n=1}^{N}\left(\frac{w_h}{\rho}\nabla_h(\frac{1}{\rho}\nabla_h u_{0,h}) + \frac{1}{\rho^2}\nabla_h w_h\nabla_h u_{0,h}\right)\,\Delta t \\ \nabla_{c,h}\tilde{J} &= \sum_{s=1}^{S}\sum_{n=1}^{N}2\frac{w_{0,h}}{c}\left(\nabla_h(\frac{1}{\rho}\nabla_h u_h) + (q_p + 2q_c)\nabla_h(\frac{1}{\rho}\nabla_h u_{0,h}) - 2\frac{w_{0,h}}{c}\nabla_h(\frac{q_p}{\rho}\nabla_h u_{0,h})\right)\,\Delta t \\ &+ \sum_{s=1}^{S}\sum_{n=1}^{N}2\frac{w_h}{c}\nabla_h(\frac{1}{\rho}\nabla_h u_{0,h})\,\Delta t \end{aligned}$$

Remark :

It is important to notice the different contribution of a variation in density, or a variation in velocity. The first one will essntially have an effect on the amplitude of the reflected signals, whereas the second one will have an effect on the kinematic of the different arrivals. Therefore a variation in velocity will be much more non linear effect on the cost function than a variation in density.

That is why, thinking of the computational cost of gradients, it would be interesting to drop the computation of the gradient with respect to ρ, and account for the variation of amplitude in the 'inner' variable r_ρ.

5 Appendix

5.1 Appendix 1

The coefficients β_l are defined by $\beta_l = \alpha_l/(2l-1)$. For consistency reasons α_l are solutions of

$$\begin{aligned} \sum_{l=1}^{L} \alpha_l &= 1 \\ \sum_{l=1}^{L} (2l-1)^{2p} \alpha_l &= 0 \qquad p = 1..L \end{aligned}$$

Solving this linear system gives :

$$\alpha_l = (-1)^{l+1} \frac{\prod_{m \neq l}^{L} (2m-1)}{\prod_{m \neq l}^{L} |(2m-1)^2 - (2l-1)^2|}$$

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