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This paper describes a technique for generating standard bilinear programming test problems with known solutions and properties. The proposed construction technique applies a simple random tranformation of variables to a separable bilinear programming problem that is constructed by combining disjoint low-dimensional bilinear programs.

Key Words and Phrases: bilinear programming, test problem generation, indefinite quadratic programming, bimatrix games.

1 Introduction.

A standard bilinear program in the variables x and y is an NP-hard problem that involves the minimization of a nonconvex quadratic objective that is void of pure quadratic terms in either x or y, over a feasible domain defined by two polyhedral sets - one in x and the other in y. The fact that these problems are NP-hard follows from their equivalence with concave quadratic programs [Kon76b] and from the NP-hard property of these QP problems [Sah74].

The many applications of bilinear programming that have been reported in the literature [Kon71, Vai74] are evidence of the importance of these problems. It is therefore no surprise that several techniques have been developed for solving them. Various cutting plane algorithms [Kon76a, SS80, VS77] are the most common of these techniques. Less common approaches include the polynomial annexation strategy proposed by Vaish and Shetty [VS76] and an approach that hinges on an equivalence between bilinear programs and linear bilevel programming problems. Gallo and Ülkücü [GU77] exploited this equivalence, without apparent knowledge of bilevel programming, to develop a solution technique based on branch and bound methods. Recently, a technique that applies sequential LCP methods [JF91] has been proposed for solving bilinear programs, and a variant of the parametric simplex algorithm [YK91] has been proposed for solving rank two and rank three bilinear programs.

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To date, all the bilinear programming test problems that have been reported in the literature have been generated randomly, or are special instances of concave quadratic programming test problems (see [Kon76a] for example). We know of no technique specifically designed to construct standard bilinear programming test problems with know solutions and properties. The purpose of this paper is to propose such a technique.

In section 2 we give a mathematical formulation of the standard bilinear programming problem and outline our approach for constructing these problems. In section 3 we describe two special bilinear programs that are used in constructing the separable bilinear program presented in section 4. The equivalence between this separable bilinear program, with known solutions and properties, and the random bilinear programs that result from a simple nonsingular transformation of variables, is also established in section 4. Section 5 presents our concluding remarks.

2 Problem Definition and Motivation.

The *standard* bilinear programming problem, problem BP, can be defined as follows:

$$\text{minimize } f(x,y) = c^T x + x^T Q y + d^T y$$

subject to

$$Ax \leq a, \quad By \leq b$$

where $c, x \in R^n$, $d, y \in R^m$, $Q \in R^{n\times m}$, $A \in R^{\alpha\times n}$, $B \in R^{\beta\times m}$, $a \in R^\alpha$ and $b \in R^\beta$.

Problem BP exhibits the following two properties:

Property 2.1 *If problem BP has a finite optimal solution then there exists an extreme point $\bar{x}$ of the polyhedron $X = \{x \in R^n : Ax \leq a\}$ and an extreme point $\bar{y}$ of the polyhedron $Y = \{y \in R^m : By \leq b\}$ such that $(\bar{x}, \bar{y})$ is an optimal solution of problem BP.*

Proof. See [Kon76a]. □

Property 2.2 *If $(\bar{x}, \bar{y})$ is a local minimum of problem BP then $\bar{x}$ is an optimal solution of the linear program:*

$$\min_{x\in X} x^T(c + Q\bar{y})$$

and $\bar{y}$ is an optimal solution of the linear program:

$$\min_{y\in Y} y^T(d + Q^T\bar{x}).$$

In other words, $\bar{x} = \arg\min_{x\in X} f(x,\bar{y})$ and $\bar{y} = \arg\min_{y\in Y} f(\bar{x}, y)$.

Proof. If $(\bar{x}, \bar{y})$ is a local minimum of problem BP then it is a stationary point of the indefinite quadratic program:

$$\min_{x\in X,\, y\in Y} \begin{bmatrix} c \\ d \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 0 & Q \\ Q^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Hence

$$\begin{bmatrix} x - \bar{x} \\ y - \bar{y} \end{bmatrix}^T \left(\begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} 0 & Q \\ Q^T & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \right) \geq 0, \quad \forall (x,y) \in X \times Y. \qquad (1)$$

Setting $y = \bar{y}$ in (1) yields:

$$\begin{bmatrix} x - \bar{x} \\ 0 \end{bmatrix}^T \begin{bmatrix} c + Q\bar{y} \\ d + Q^T\bar{x} \end{bmatrix} \geq 0, \ \ \forall x \in X.$$

Therefore, $\forall x \in X$, $x^T(c + Q\bar{y}) \geq \bar{x}^T(c + Q\bar{y})$ and this establishes the first assertion.

The second assertion is proved in a similar manner by setting $x = \bar{x}$ in (1). □

It is our intention to construct standard bilinear programming problems by performing a random nonsingular transformation of variables to a separable version of problem BP. This separable version of problem BP, which is described in section 4, is constructed by combining disjoint low-dimensional bilinear programs. These kernel programs, which are described in the following section, are constructed in such a way that their solutions are known as a consequence of properties 2.1 and 2.2.

3 Kernel Bilinear Programs.

In this section we describe two sets of kernel bilinear programs. These two sets will be combined to generate a separable bilinear programs which, in turn, will be used to construct standard bilinear programming test problems.

Kernel Program 1

For $k = 1, \ldots, \kappa_1$ define problem BP_k to be the following four-variable two-parameter bilinear program:

$$\text{minimize } f_k(x_k, y_k) = {c_k}^T x_k + {x_k}^T Q_k y_k + {d_k}^T y_k$$

subject to

$$A_k x_k \leq a_k, \ \ B_k y_k \leq b_k$$

where $x_k, y_k \in R^2$,

$$c_k = d_k = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \text{ and } Q_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_k = \begin{bmatrix} 0 & 1 \\ -2 & -1 \\ 2 & -1 \end{bmatrix}, a_k = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, B_k = \begin{bmatrix} -\delta_k & 1 \\ \delta_k - \rho_k & 1 \\ \rho_k & -2 \end{bmatrix} \text{ and } b_k = \begin{bmatrix} 0 \\ 2\delta_k - \rho_k \\ 0 \end{bmatrix}.$$

The following special classes of problem BP_k, $k \in \{1, \ldots, \kappa_1\}$, will be used in constructing a separable version of problem BP:

- Class 1: $1 \leq \delta_k < 3$ and $\rho_k = 0$.
- Class 2: $\delta_k = 3$ and $\rho_k = 0$.
- Class 3: $\delta_k > 3$ and $\rho_k = 0$.
- Class 4: $\delta_k = 5/2$ and $\rho_k = 3/2$.

For each of these problem classes the sets $X_k = \{x \in R^2 : A_k x \leq a_k\}$ and $Y_k = \{y \in R^2 : B_k y \leq b_k\}$ define polytopes. Consequently, by virtue of property 2.1, these problems will have a global minimum $(\bar{x}, \bar{y})$ with $\bar{x}$ an extreme point of X_k and $\bar{y}$ an extreme point of Y_k. In addition, each of these problem classes exhibits different solution properties as a result of their distinct constraint geometries. These properties are presented below.

Property 3.1 *For class 1 problems (see figures 1(a) and 1(b)), problem BP_k has exactly four local minima; namely, the three extreme points $(x_k^{(1)}, y_k^{(1)}) = (0,2,2,0)$, $(x_k^{(2)}, y_k^{(2)}) = (2,2,0,0)$ and $(x_k^{(3)}, y_k^{(3)}) = (1,0,1,\delta_k)$, and the nonextreme point $(x_k^{(4)}, y_k^{(4)}) = (1,2,1,0)$. The first two points are global minima with $f_k(x_k^{(l)}, y_k^{(l)}) = -4$, $l = 1,2$, whereas $f_k(x_k^{(3)}, y_k^{(3)}) = -(\delta_k+1)$ and $f_k(x_k^{(4)}, y_k^{(4)}) = -3$.*

Proof. For $l = 1,2,3$ and 4, $x_k^{(l)} = \arg\min_{x \in X_k} f_k(x, y_k^{(l)})$ and $y_k^{(l)} = \arg\min_{y \in Y_k} f_k(x_k^{(l)}, y)$. In other words $(x_k^{(l)}, y_k^{(l)})$, $l = 1,2,3,4$, satisfy the necessary conditions of property 2.2. The local optimality of these points is established by noting that all feasible directions from these points are either strict ascent directions or stationary directions of nonnegative curvature. Finally, the exclusivity of these four points follows since no other points satisfy property 2.2. □

Similar arguments to those given above can be used to establish the following properties pertaining to the remaining three classes.

Property 3.2 *For class 2 and 3 problems the points $(x_k^{(1)}, y_k^{(1)}) = (0,2,2,0)$, $(x_k^{(2)}, y_k^{(2)}) = (2,2,0,0)$, $(x_k^{(3)}, y_k^{(3)}) = (1,0,1,\delta_k)$ and $(x_k^{(4)}, y_k^{(4)}) = (1,2,1,0)$ are the only local minima of problem BP_k. For class 2 problems, the first three of these points are global minima with function value $f_k(x_k^{(l)}, y_k^{(l)}) = -4$, $l = 1,2,3$, whereas, for class 3 problems, the point $(x_k^{(3)}, y_k^{(3)}) = (1,0,1,\delta_k)$ is the unique global minimum with $f_k(x_k^{(3)}, y_k^{(3)}) = -(\delta_k + 1)$.*

Property 3.3 *For class 4 problems (see figures 1(a) and 1(c)) the point $(x_k^{(2)}, y_k^{(2)}) = (2,2,0,0)$ is the unique global minimum of problem BP_k with function value $f_k(x_k^{(2)}, y_k^{(2)}) = -4$. The point $(x_k^{(3)}, y_k^{(3)}) = (1,0,1,\delta_k)$ is the only (other) local minima with $f_k(x_k^{(3)}, y_k^{(3)}) = -(\delta_k + 1)$.*

Kernel Program 2

For $k = \kappa_1 + 1, \ldots, \kappa_2$ define problem BP_k to be the following three-variable bilinear program:

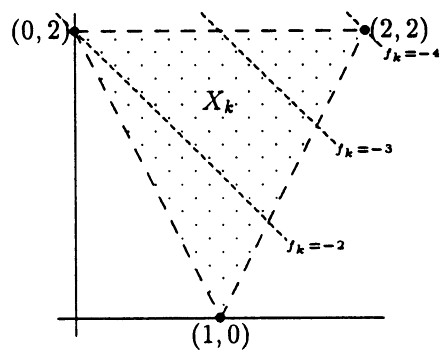
$$\text{minimize } f_k(x_k, y_k) = {c_k}^T x_k + {x_k}^T Q_k y_k + d_k y_k$$

subject to

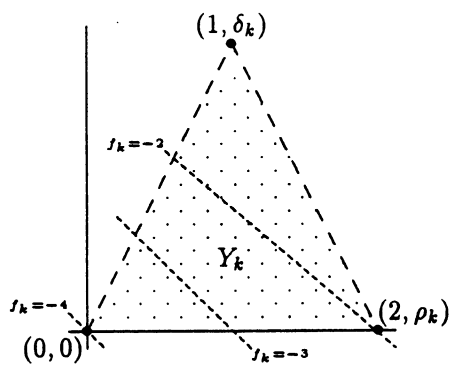
$$A_k x_k \leq a_k, \quad B_k y_k \leq b_k$$

where $x_k \in R^2$, $y_k \in R$,

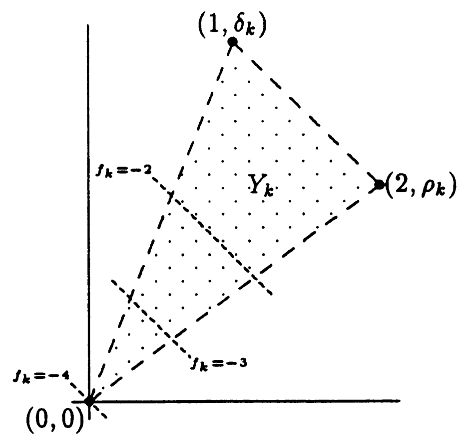
$$c_k = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, d_k = -2 \text{ and } Q_k = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$



(a) X_k and contours of $f_k(x, y_k^{(2)})$.



(b) Y_k and contours of $f_k(x_k^{(2)}, y)$, $\delta_k = 2$, $\rho_k = 0$.



(c) Y_k and contours of $f_k(x_k^{(2)}, y)$, $\delta_k = 5/2$, $\rho_k = 3/2$.

Figure 1: X_k, Y_k and contours of f_k for class 1 (figures (a) and (b)) and class 4 (figures (a) and (c)) problems.

$$A_k = \begin{bmatrix} 0 & 1 \\ -2 & -1 \\ 2 & -1 \end{bmatrix}, a_k = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, B_k = 1 \text{ and } b_k = 2.$$

For these programs the set $X_k = \{x \in R^2 : A_k x \le a_k\}$ defines a polytope but the set $Y_k = \{y \in R^2 : B_k y \le b_k\}$ now defines a polyhedron. However property 2.1 still applies (ie. problem BP has a finite solution). In fact, using arguments similar to those given for kernel program 1, we are able to make the following observation about the solution to these problems:

Property 3.4 *For $k = \kappa_1 + 1, \ldots, \kappa_2$ problem BP_k has the unique minimum $(x_k^{(5)}, y_k^{(5)}) = (1, 0, 2)$, with function value $f_k(x_k^{(5)}, y_k^{(5)}) = -3$, and no other local minima.*

In the next section we describe the bilinear test problems that result when these kernel programs are combined and then randomly transformed.

4 Constructing Random Bilinear Programs.

The observations from the last section guarantee the following:

Proposition 4.1 *If $(\bar{x}_k, \bar{y}_k)$ is a minimum of problem BP_k, for $k = 1, \ldots, \kappa_2$, then $(\bar{x}_1 \cdots \bar{x}_{\kappa_2} \bar{y}_1 \cdots \bar{y}_{\kappa_2})$ is a minimum of the following separable bilinear program:*

$$\textit{minimize } f(x, y) = \sum_{k=1}^{\kappa_2} f_k(x_k, y_k)$$

subject to

$$x_k \in X_k, \quad y_k \in Y_k \qquad k = 1, \ldots, \kappa_2.$$

This separable bilinear program can be rewritten, as problem $BP(c, d, Q, A, a, B, b)$:

$$\text{minimize } f(x, y) = c^T x + x^T Q y + d^T y$$

subject to

$$Ax \le a, \quad By \le b$$

where $x \in R^{2\kappa_2}$, $y \in R^{\kappa_1 + \kappa_2}$,

$$c^T = [c_1^T \cdots c_{\kappa_2}^T], d^T = [d_1^T \cdots d_{\kappa_2}^T] \text{ and } Q = \text{diag}(Q_1 \cdots Q_{\kappa_2}),$$

$A = \text{diag}(A_1 \cdots A_{\kappa_2}), a^T = [a_1^T \cdots a_{\kappa_2}^T], B = \text{diag}(B_1 \cdots B_{\kappa_2})$ and $b^T = [b_1^T \cdots b_{\kappa_2}^T]$.

Based on proposition 4.1 the following can be said about the minima of problem $BP(c, d, Q, A, a, B, b)$:

Corollary 4.1 *Problem $BP(c, d, Q, A, a, B, b)$ has $4^{\kappa_{1,1}+\kappa_{1,2}+\kappa_{1,3}} \cdot 2^{\kappa_{1,4}}$ local minima including $2^{\kappa_{1,1}} \cdot 3^{\kappa_{1,2}}$ global minima, where $\kappa_{1,i}$, $i = 1, 2, 3, 4$, is the cardinality of the set $\{k \le \kappa_1 :$ problem BP_k is in Class $i\}$.*

For $n_x = 2\kappa_2$ and $n_y = \kappa_1 + \kappa_2$ define the order-n_x matrix $M_x = D_x H_x$ and the order-n_y matrix $M_y = D_y H_y$, where H_x and H_y are random Householder matrices and D_x and D_y are positive definite diagonal matrices. In addition, let $W_x = M_x^{-1} = H_x D_x^{-1}$ and $W_y = M_y^{-1} = H_y D_y^{-1}$. With these definitions we can construct the random bilinear problem $BP(\bar{c}, \bar{d}, \bar{Q}, \bar{A}, a, \bar{B}, b)$, where $\bar{c} = M_x^T c$, $\bar{d} = M_y^T d$, $\bar{Q} = M_x^T Q M_y$, $\bar{A} = A M_x$ and $\bar{B} = B M_y$.

The relationship between separable problem $BP(c, d, Q, A, a, B, b)$ and this transformed problem is characterized by the following result:

Proposition 4.2 *Problem $BP(c,d,Q,A,a,B,b)$ in variables $x \in R^{n_x}$ and $y \in R^{n_y}$ is equivalent to problem $BP(\bar{c},\bar{d},\bar{Q},\bar{A},a,\bar{B},b)$ in the variables $\bar{x} \in R^{n_x}$ and $\bar{y} \in R^{n_y}$ under the nonsingular transformations $\bar{x} = W_x x$ and $\bar{y} = W_y y$.*

Proof. For $x = M_x\bar{x}$ and $y = M_y\bar{y}$ problem $BP(c,d,Q,A,a,B,b)$ becomes

$$\text{minimize } f(M_x\bar{x}, M_y\bar{y}) = c^T M_x\bar{x} + \bar{x}^T(M_x^T Q M_y)\bar{y} + d^T M_y\bar{y}$$

subject to

$$[AM_x]\bar{x} \le a, \ [BM_y]\bar{y} \le \bar{y}$$

which is problem $BP(\bar{c},\bar{d},\bar{Q},\bar{A},a,\bar{B},b)$ in the variables $\bar{x} \in R^{n_x}$ and $\bar{y} \in R^{n_y}$.
□

Corollary 4.2 *If (x,y) is a local minimum of problem $BP(c,d,Q,A,a,B,b)$ then $(W_x x, W_y y)$ is a local minimum of problem $BP(\bar{c},\bar{d},\bar{Q},\bar{A},a,\bar{B},b)$, where $\bar{c} = M_x^T c$, $\bar{d} = M_y^T d$, $\bar{Q} = M_x^T Q M_y$, $\bar{A} = AM_x$ and $\bar{B} = BM_y$. Similarly, if $(\bar{x},\bar{y})$ is a local minimum of problem $BP(\bar{c},\bar{d},\bar{Q},\bar{A},a,\bar{B},b)$ then $(M_x\bar{x}, M_y\bar{y})$ is a local minimum of problem $BP(c,d,Q,A,a,B,b)$.*

As a consequence of proposition 4.2 and corollary 4.2 it is apparent that kernel programs 1 and 2 can be used to construct a separable bilinear program (with known minima) which can be transformed to generate (equivalent) random bilinear programming test problems. In addition, the sparsity of the data that defines problem $BP(\bar{c},\bar{d},\bar{Q},\bar{A},a,\bar{B},b)$ can be adjusted by controlling the sparsity of the Householder vectors that generate H_x and H_y, and the spectrum of $M_x^T Q M_y$ (or the geometry of problem $BP(\bar{c},\bar{d},\bar{Q},\bar{A},a,\bar{B},b)$) can be adjusted by controlling the entries of D_x and D_y.

5 Concluding Remarks.

Several applications of bilinear programs, and various solution techniques for these problems, have been reported in the literature. To test and improve these individual solution techniques, and to make valid comparisons between different techniques, requires a suite of test problems. This paper describes a simple but effective technique for constructing such problems. The properties and solutions of the constructed problems can be controlled by the user.

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