A Trace Theorem for Solutions of Linear Partial Differential Equations

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CRPC-TR89017
October 1989

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Introduction

The standard trace theorem in *Sobolev space* $H^s(\mathbb{R}^n)$ indicates that the trace map which denotes the restriction of each distribution to a codimension one hypersurface extends uniquely to a continuous linear operator from $H^s(\mathbb{R}^n)$ to $H^{s-1/2}(\mathbb{R}^n)$, if $s > 1/2$, see Taylor [13] or Hörmander [6] for details. It is also well known that this trace theorem is sharp. However, it seems quite natural that one may expect an improvement of regularity of the trace if the distribution is a solution of a linear partial differential equation. Obviously, the optimal case will be that of no loss (or even a gain) of smoothness. The

[1]This work was partially supported by the National Science Foundation under grant DMS 86-03614 and DMS 89-05878, and by the Office of Naval Research under contracts N00014-K-85-0725 and N00014-J-89-1115.

main goal of this work is to determine circumstances under which the trace of the solution of a linear partial differential equation is as smooth as the solution itself.

The trace properties of solutions to linear partial differrential equations have been used widely in various problems such as boundary value problems, initial-boundary value problems, control problems and in particular, many inverse problems. In [10], Symes proved a trace theorem for a second order multidimensional 3 compactly supported away from the boundary, the trace is of class H^1_{loc} which is the same regularity class as the solution in the interior. His examples, in the same article, also showed that additional smoothness of initial data along certain directions (corresponding to grazing rays) is necessary for the trace to be so regular. We refer to Symes[11] for some more comments, which turn out to be the original idea of our work here.

Clearly, if the linear partial differential equation is strictly hyperbolic with smooth coefficients, standard energy estimates will yield the fact that the solution along any spacelike trace is as smooth as itself locally, provided a sufficiently smooth right-hand side. Unfortunately, for more general equations or even a strictly hyperbolic differential equation but this time along a nonspacelike trace, the same idea will not work, essentially because one does not know how to apply energy estimates to a nonhyperbolic problem directly.

In this paper, we shall investigate the trace regularities of solutions to linear *P.D.E.* Our result shows that the difficulties discussed above may be cured by imposing some additional microlocal smoothness. In order to see why this is so, let us begin with the following definition.

Definition. *A distribution u is said to be in $H^s \cap H^r_{ml}(x_0, \xi_0)$ if there exist $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(x_0) \neq 0$ and a conic neighborhood $\Gamma \subset \mathbb{R}^n \backslash \{0\}$ of ξ_0 such that*

$$\langle \xi \rangle^s (\phi u)^\wedge(\xi) \in L^2(\mathbb{R}^n)$$

and

$$\langle \xi \rangle^r \chi_\Gamma(\xi) (\phi u)^\wedge(\xi) \in L^2(\mathbb{R}^n) \, .$$

The reader is referred to Beals [2], Rauch [9] and references cited there for an overlook of microlocal analysis and its applications to nonlinear problems.

Roughly speaking, our trace theorem says that the solution will belong to H^s along a codimension one hypersurface if it belongs to H^s and to H^{s+1} in those directions where the *P.D.E* is not microlocally strictly hyperbolic.

The proof of the theorem is based on a pseudodifferential cutoff technique. Similar techniques have been used by many people, see for example, Hömander [7], also Nirenberg [8]. The main idea is to alter the *P.D.O.* microlocally to make a nice strictly hyperbolic linear pseudo-differential equation for which the trace hypersurface is spacelike, then estimate the remainder *via* a lemma stated in a fairly general form.

We believe that the trace theorem, the lemma and the techniques used in this work will be helpful in various of other contexts. Some of the applications have already been seen in our recent work of understanding multidimensional hyperbolic inverse problems with smooth or nonsmooth coefficients, which will be reported elsewhere.

Notation. Throughout htis paper, the reader is assumed to be familiar with the basic calculus of *Pseudodifferential Operators* ($\Psi.D.O.$) as stated in Taylor [13] or Nirenberg [8]. For simplicity, C serves as a generalized positive constant the precise value of which is not needed. Usually, the constant from *Fourier* transform is assumed to be absorbed by the integal. $WF(u)$ is the wavefront set of u, $ES(P)$ stands for the essential support of operator P, both $\mathcal{F}$ and $\wedge$ mean *Fourier* transform and $\langle\xi\rangle$ is $(1+ \mid \xi \mid^2)^{1/2}$.

We close this introduction by the plan of this paper. Several useful features concerning the smooth family of $\Psi.D.O$ are introduced in Section 1. the trace theorem is atated and proved in Section 2. In Section 3, we make a few comments on the trace theorem and give a half-space trace regularity result as a corollary of the trace theorem.

1 Preliminary Results

It is interesting to see that from the definition a smooth family of $\Psi.D.O.$ $P(x,y,D_x) \in OPS^r_{1,0}(\mathbb{R}^n \times \mathbb{R}^m)$ with $m \leq n$ is not a $\Psi.D.O.$ in $\mathbb{R}^n$. This phenomenon was observed by Taylor in the Appendix of [12]. Fortunately, as he also pointed out, the operator acts like a $\Psi.D.O.$ on the types of functions and distributions we are insterested in. This section is devoted to the understanding of these $\Psi.D.O.$-like operators. We begin with our Proposition 1 which guarantees that similar *Sobolev space* continuous properties still hold.

Proposition 1 *If* $p(x,\xi) \in S^r_{1,0}(\mathbb{R}^m \times \mathbb{R}^m)$, $1 \leq m \leq n$, *satisfies one of the following assumptions:*

1. $p(x,\xi) = p(\xi)$, i.e., *it is independent of* x*;*

2. $p(x,\xi)$ *has compact support in* x*,*

then

$$p(x, D_x) : H^s(\mathbb{R}^n) \to H^{s-r}(\mathbb{R}^n)$$

continuously.

Proof. For simplicity, we only prove the second statement here. The first one follows directly from the fact $\mathcal{F}[P(D)u(x)] = P(\xi)\hat{u}(\xi)$. It suffices to prove for r=0 case. Also, it suffices to derive the appropriate norm estimates. Let $u \in \mathcal{S}$, the *Schwarz* space, write $p(x,\xi) = \int \mathcal{F}_x p(\eta,\xi)e^{ix\eta}d\eta$, with $\mathcal{F}_x p(\eta,\xi) = \int p(x_1,\xi)e^{-ix_1\eta}dx_1$. Assumption 2 on $p(x,\xi)$ implies that

$$| \mathcal{F}_x p(\eta,\xi) | \leq C_N(1+ | \eta |^2)^{-N/2} , \quad \forall N > 0 ,$$

knowing that

$$\eta^\alpha \mathcal{F}_x p(\eta,\xi) = \int D^\alpha_{x_1} p(x_1,\xi)e^{-ix_1\eta}dx_1 .$$

By the definition of $\Psi.D.O.$,

$$P(x,D)u(x,y) = \int p(x,\xi)\mathcal{F}_x u(\xi,y)e^{ix\xi}d\xi .$$

Taking the *Fourier* transform on both sides, we have

$$\begin{aligned} \mathcal{F}(P(x,D)u)(\eta,\zeta) &= \int\int\int P(x,\xi)\mathcal{F}_x u(\xi,y)e^{ix\xi-ix\eta-iy\xi}d\xi dx dy \\ &= \int \mathcal{F}_x P(\eta-\xi,\xi)\hat{u}(\xi,\zeta)d\xi , \end{aligned}$$

hence

$$| \mathcal{F}(P(x,D)u)(\eta,\zeta) | \leq C_N \int (1+ | \eta - \xi |^2)^{-N/2} | \hat{u}(\xi,\zeta) | \, d\xi \, .$$

Therefore,

$$\begin{aligned} \| P(x,D)u \|_{H^s}^2 &= \| (1+ | \eta |^2 + | \zeta |^2)^{s/2} | \mathcal{F}(P(x,D)u)(\eta,\zeta) | \|_{L^2(\eta,\zeta)}^2 \\ &\leq C \left\| \int (1+ | \eta |^2 + | \zeta |^2)^{s/2} (1+ | \eta - \xi |^2)^{-N/2} | \hat{u}(\xi,\zeta) | d\xi \right\|_{L^2(\eta,\zeta)}^2 , \end{aligned}$$

then, the Hörmander inequality yields

$$\begin{aligned} &\| P(x,D)u \|_{H^s}^2 \\ &\leq C \left\| \int (1+ | \eta - \xi |^2)^{-N_1/2} (1+ | \xi |^2 + | \zeta |^2)^{s/2} | \hat{u}(\xi,\zeta) | d\xi \right\|_{L^2(\eta,\zeta)}^2 \\ &= C \int I d\zeta \, , \end{aligned}$$

with $N_1 = N - s$, and

$$I = \left\| \int (1+ | \eta - \xi |^2)^{-N_1/2} (1+ | \xi |^2 + | \zeta |^2)^{s/2} | \hat{u}(\xi,\zeta) | d\xi \right\|_{L^2(\eta)}^2 .$$

Using Young's inequality and the fact

$$(1+ | \eta - \xi |^2)^{-N_1/2} \in L^1(\eta) \cap L^1(\xi) \, ,$$

for large N_1, we have

$$I \leq C \| (1+ | \xi |^2 + | \zeta |^2)^{s/2} | \hat{u}(\xi,\zeta) | \|_{L^2(\xi)}^2 .$$

Finally, we obtain that

$$\begin{aligned}
\| P(x,D)u \|_{H^s}^2 &\leq C \int \| (1 + | \xi |^2 + | \zeta |^2)^{s/2} | \hat{u}(\xi,\zeta) | \|_{L^2(\xi)}^2 d\zeta \\
&\leq C \| (1 + | \xi |^2 + | \zeta |^2)^{s/2} | \hat{u}(\xi,\zeta) | \|_{L^2(\xi,\zeta)}^2 \\
&\leq C \| u \|_{H^s}^2 .
\end{aligned}$$

□

Note that, the only thing prevents $P(x, y, D_x)$ from being a $\Psi.D.O.$ of order r is that its symbol $p(x, y, \xi)$ does not decrease in any directions other than ξ-direction. This implies that *via* a pseudodifferential cutoff along those nondecay directions P may be regularized to be a $\Psi.D.O.$, which leads to our next proposition.

Proposition 2 *Assume that* $P \in OPS^r_{1,0}(\mathbb{R}^n \times \mathbb{R}^m)$, $H \in OPS^s_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $1 \leq m \leq n$, $r, s \in \mathbb{R}$ *and* $\phi(x, y) \in C_0^\infty(\mathbb{R}^n)$. *Furthermore, assume*

$$ES(H) \subset \Gamma ,$$

where Γ *is a (closed) conic neighborhood of*

$$\{(x, y, \xi, \eta) \in (\mathbb{R}^n \times \mathbb{R}^n),\ (\xi_{m+1}, ..., \xi_n) = 0\} .$$

Then

$$P\phi H \in OPS^{r+s}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) .$$

Proof. W.L.O.G. it is sufficient to show that

$$Q = P(x,y,D_x)\phi(x,y)H(D_x,D_y)$$

is a $\Psi.D.O$ (in $OPS^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$). Observe that

$$P\phi Hu(x,y) = \int P(x,y,\xi)e^{ix\xi}\mathcal{F}(\phi Hu)(\xi,y)d\xi$$

and

$$\begin{aligned}
\mathcal{F}(\phi Hu)(\xi,y) &= \int \phi(x',y)H(D_x,D_y)u(x',y)e^{-ix'\xi}dx', \\
H(D_x,D_y)u(x',y) &= \int\int H(\xi',\eta')\hat{u}(\xi',\eta')e^{ix'\xi'+iy\eta'}d\xi'd\eta',
\end{aligned}$$

then

$Qu(x,y)$

$$\begin{aligned}
&= \int\int\int\int P(x,y,\xi)e^{ix\xi}\phi(x',y)e^{-ix'\xi}H(\xi',\eta')\hat{u}(\xi',\eta')e^{ix'\xi'+iy\eta'}d\xi'd\eta'dx'd\xi \\
&= \int\int[\int\int P(x,y,\xi)\phi(x',y)e^{ix(\xi-\xi')-ix'(\xi-\xi')}dx'd\xi] \\
&\quad H(\xi',\eta')\hat{u}(\xi',\eta')e^{ix'\xi'+iy\eta'}d\xi'd\eta' \\
&= \int\int[\int P(x,y,\xi)\hat{\phi}(\xi-\xi',y)e^{ix(\xi-\xi')}d\xi]H(\xi',\eta')\hat{u}(\xi',\eta')e^{ix\xi'+iy\eta'}d\xi'd\eta',
\end{aligned}$$

hence, the symbol of Q

$$\begin{aligned}
Q_0 &= \int P(x,y,\xi)\hat{\phi}(\xi-\xi',y)e^{ix(\xi-\xi')}d\xi H(\xi',\eta') \\
&= \int P(x,y,\xi+\xi')\hat{\phi}(\xi,y)e^{ix\xi}d\xi H(\xi',\eta')\,.
\end{aligned}$$

To show that $Q \in OPS^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, it suffices to prove

$$\mid \partial^{\alpha}_{\xi'} Q_0(x,y,\xi',\eta') \mid \leq C_{\alpha,K}(1+ \mid \xi' \mid + \mid \eta' \mid)^{-\alpha},$$

for any $(\xi',\eta') \in \mathbb{R}^n$, $(x,y) \in K$ (a compact set in $\mathbb{R}^n$).

But, from definition,

$$\begin{aligned} \mid \partial^{\alpha_1}_{\xi'} P(x,y,\xi+\xi') \mid &\leq C_{\alpha,K}(1+ \mid \xi+\xi' \mid)^{-\alpha_1} \\ &\leq C(1+ \mid \xi' \mid)^{-\alpha_1}(1+ \mid \xi \mid)^{-\alpha_1}, \ \forall \alpha_1 > 0\,, \end{aligned}$$

it follows that

$$\begin{aligned} &\mid \partial^{\alpha}_{\xi'} Q_0(x,y,\xi',\eta') \mid \\ &\leq C \mid \sum_{0 \leq \alpha_1 \leq \alpha} [\partial^{\alpha_1}_{\xi'} \int P(x,y,\xi+\xi') e^{ix\xi} \hat{\phi}(\xi,y) d\xi] \ \partial^{\alpha-\alpha_1}_{\xi'} H(\xi',\eta') \mid \\ &\leq \sum_{0 \leq \alpha_1 \leq \alpha} C(1+ \mid \xi' \mid)^{-\alpha_1} \partial^{\alpha-\alpha_1}_{\xi'} H(\xi',\eta') \\ &\leq C_{\alpha,K}(1+ \mid \xi' \mid + \mid \eta' \mid)^{-\alpha}, \end{aligned}$$

where the last inequality comes from our construction of H, *i.e.* $H(\xi',\eta')$ nonzero only in the region $(1+ \mid \xi' \mid + \mid \eta' \mid) \leq C(1+ \mid \xi' \mid)$. □

2 Trace Theorem

We can now state and prove the main result in this paper, a trace theorem.

¿From now on, t will serve as a distinguished variable.

Theorem. *Assume that*

$$P(x,t,\xi,\tau) = \tau^m + \sum_{j=0}^{m-1} a_j(x,t)\tau^j\xi^{m-j}$$

is the principal symbol of a linear partial differential operator $P(x,t,D_x,D_t)$ *with uniformly bounded smooth coefficients* $\{a_j(x,t)\}$.

Let $\Omega \subset\subset \mathbb{R}^n = \{t=0\}$, *and* Υ *=closed conic set* $\subset T^*(\mathbb{R}^n)|_\Omega$ *such that* $(x,\xi) \in \Upsilon \Rightarrow P(x,0,\xi,\tau)$ *has* m *distinct real roots as a polynomial of* τ.

Let Γ *be a closed conic neighborhood of*

$$\Gamma_{C_0} = \{(x,t,\xi,\tau) \in T^*(\mathbb{R}^{n+1}), \tau^2/(1+\sum_{j=1}^{n} |\xi_j|^2) \le C_0 < +\infty\},$$

i.e., Γ *is closed and does not intersect with the normal bundle of* $\{t=0\}$.

Also, assume that u *satisfies the equation* $Pu=0$ *and*

$$u \in H^s \bigcap H^{s+1}_{ml}(\Gamma \bigcap \Upsilon'),$$

where $\Upsilon' = \Pi^*(\Upsilon_1) \subset T^*(\mathbb{R}^{n+1})$*, and* Υ_1 *is a conic neighborhood of* Υ^C*.*

Then, for $\phi(x,t) \in C_0^\infty(\Omega_1)$ *with* $\Omega_1 \subset\subset \mathbb{R}^{n+1}$*,* $\Omega_1 \cap \{t=0\} \subset \Omega$*,*

$$\phi u \mid_{t=0} \in H^s.$$

Remark. If P is elliptic, the stronger conclusion holds by the classical trace theorem. On the other hand, if P is strictly hyperbolic with respect to the trace $\{t = 0\}$, the conclusion also holds from standard hyperbolic energy estimates.

Proof of Theorem. For the situations where the operator P is neither elliptic nor strictly hyperbolic, i.e. both Υ and Υ^C are not empty, the idea of proof is to construct a strictly hyperbolic Cauchy problem by a $\Psi.D.O$ cutoff. Since the problem is local, one may assume that $\Upsilon = \Omega \times \mathcal{U}$ and $\Upsilon_1 = \Omega \times \mathcal{U}_1$, where $\mathcal{U}$ is a conic subset of $\mathbb{R}^n$ and $\mathcal{U}_1$ is a conic neighborhood of $\mathcal{U}^C$, which guarantee that we can find a convolutional $\Psi.D.O$ $Q \in OPS^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ ($i.e.$ $Q(x,\xi) = Q(\xi)$) such that

- $ES(Q) \subset$ the conic neighborhood $\mathcal{U}_1$ of $\mathcal{U}^C$;
- $Q_0 = 1$ in $\mathcal{U}^C$, $Q_0 = 0$ in ${\mathcal{U}_1}^C$ and $0 \le Q_0 \le 1$,

where Q_0 is the principal symbol of Q. Then, consider

$$\tilde{P} = (I-Q)P + QP_0 \stackrel{\text{def}}{=} (I-Q)P + Q\prod_{i=1}^{m}(\tau - \alpha_i \mid \xi \mid) ,$$

with $\{\alpha_i\}$ are real and distinct constants. It follows that $\tilde{P}$ is strictly hyperbolic $\Psi.D.O.$ of order m and diferential in t. Now we have the following strictly hyperbolic Cauchy problem of ϕu (recall ϕ has compact support)

$$\begin{aligned}
\tilde{P}\phi u &= (I-Q)P\phi u + QP_0\phi u \\
&= (I-Q)[P,\phi]u + QP_0\phi u .
\end{aligned}$$

Hence, standard hyperbolic energy estimates (see Courant and Hilbert [5] or Taylor[13]) yield

$$\begin{aligned}
\| \phi u \mid_{t=0} \|_s &\leq C \| (I-Q)[P,\phi]u + QP_0\phi u \|_{s-(m-1)} \\
&\leq C\{\| \phi_0 u \|_s + \| QP_0\phi u \|_{s-(m-1)}\} \\
&\leq C\{\| \phi_0 u \|_s + \| [Q,P_0]\phi u \|_{s-(m-1)} + \| P_0 Q\phi u \|_{s-(m-1)}\} \\
&\leq C\{\| \phi_0 u \|_s + \| [Q,P_0]\phi u \|_{s-(m-1)} + \| Q\phi u \|_{s+1}\} ,
\end{aligned}$$

where $\phi_0 \in C_0^\infty(\Omega_1)$ and $supp(\phi_0) \subseteq supp(\phi)$. Since $[Q,P_0] \in OPS_{1,0}^{m-1}$, in order to complete our proof it suffices to show that

$$Q\phi u \in H_{loc}^{s+1}(\mathbb{R}^{n+1}) ,$$

which requires the following lemma.

Lemma. *Let* $B \in OPS^r_{1,0}(\mathbb{R}^{m_0} \times \mathbb{R}^{m_0})$ *and* $A \in OPS^s_{1,0}(\mathbb{R}^{n_0} \times \mathbb{R}^{n_0})$*, where* $1 \le m_0 \le n_0$*. Let* Γ *be a conic neighborhood of*

$$\Gamma_C = \{(x,\xi) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_0}, [(\sum_{i=m_0+1}^{n_0} \mid \xi_i \mid^2)/(1 + \sum_{j=1}^{m_0} \mid \xi_j \mid^2)] \le C < +\infty\}.$$

Furthermore, assume that

1. *A is microlocal elliptic in a conic neighborhood* $\mathcal{K}$ *of* $\mathbb{R}^{2n_0} \backslash \Gamma$;

2. $u \in H^l \cap H^{l+1}_{ml}(\Gamma \cap ES(B)')$, $ES(B)' = \Pi^{-1}(ES(B))$, $l \in \mathbb{R}$;

3. $A\phi u \in H^{l-s+1}_{loc}(\mathbb{R}^{n_0})$, $\phi(x) \in C^\infty_0(\mathbb{R}^{n_0})$.

Then

$$B\phi u \in H^{l-r+1}_{loc}(\mathbb{R}^{n_0}).$$

To be able to apply the above lemma to our proof of the trace theorem, the assumptions stated in the lemma must be verified. Notice that assumption 2 is just our assumption in the theorem. Assumption 3 is easy to be verified. Since the coefficients $\{a_j(x,t)\}$ are uniformly bounded,

$$P(x,t,\xi,\tau) = \tau^m + \sum_{j=0}^{m-1} a_j(x,t)\tau^j\xi^{m-j}$$

will be microlocal elliptic in a conic neighborhood $\mathcal{K}$ of $\mathbb{R}^{2(n+1)}\backslash\Gamma_{C_0}$ if C_0 is properly chosen (sufficiently large), this verifies the assumption 1 of the lemma. Hence, as a consequence of the lemma, we obtain immediately that

$$Q\phi u \in H^{s+1}_{loc}$$

which completes the proof of our trace theorem. $\square$

Remark on the Lemma. The operator A plays a very important role here. Fortunately, for many problems the operator in the linear partial differential equation can be often chosen as A, which is implied by assumption 3.

Proof of Lemma. W.L.O.G., the proof may be reduced to the case $r = 0$.

We construct a $\Psi.D.O$ $H \in OPS^0_{1,0}(\mathbb{R}^{n_0} \times \mathbb{R}^{n_0})$ which satisfies

- $ES(H) \subseteq \Gamma$;
- $H = 1$ in $\Gamma_0 \subset \Gamma$,

where $\mathbb{R}^{2n_0}\backslash\Gamma_0$ is another conic neighborhood of $\mathbb{R}^{2n_0}\backslash\Gamma$ and contained in $\mathcal{K}$. Write $\phi = \phi\phi_1$ with $\phi_1 \in C_0^\infty(\mathbb{R}^{n_0})$, then

$$B\phi u = B\phi_1 H\phi u + B\phi_1(I - H)\phi u.$$

Since

$$A(I-H)\phi u = [A, I-H]\phi u + (I-H)A\phi u$$

and $[A, I - H]$ has order $s - 1$, we have

$$A(I - H)\phi u \in H_{loc}^{l-s+1} ,$$

follows by assumptions 1&2 and *Proposition 1.*

Applying Prop.1.10 of Chapter5 in Taylor[13], assumption 1 and the fact

$$ES(I - H) \subseteq \mathbb{R}^{2n_0} \backslash \Gamma_0 \subset \mathcal{K} ,$$

i.e., microlocal elliptic directions of A, we get

$$(I - H)\phi u \in H_{loc}^{l+1} .$$

Thus

$$B\phi_1(I - H)\phi u \in H_{loc}^{l-r+1}.$$

On the other hand, from the construction of H, our *Proposition 2* implies that

$$B\phi_1 H \in OPS_{1,0}^0(\mathbb{R}^{n_0}).$$

Moreover,

$$ES(B\phi_1 H) \subseteq ES(B)' \bigcap ES(H) \subseteq \Gamma \bigcap ES(B)'.$$

Assumption 2 allows us to write

$$u = u_1 + u_2$$

such that

$$u_1 \in H^{l+1} \text{ and } (ES(B)' \bigcap \Gamma) \bigcap WF(u_2) = \emptyset \, .$$

Thus, a simple property of wavefront set gives

$$B\phi_1 H\phi u_2 \in C^\infty \, ,$$

which yields

$$B\phi_1 H\phi u_2 \in H^{l+1}_{loc} \, .$$

Eventually, combining the above arguments, we have

$$B\phi u = B\phi_1 H\phi u + B\phi_1 (I - H)\phi u \in H^{l+1}_{loc}$$

which finishes the proof. □

3 Concluding Remarks

In this section several remarks on the trace theorem will be made. In particular, a useful corollary will be introduced.

(a). The equation is not necessarily homogeneous. Also, there are no more difficulties if the operator has smooth lower order terms. It is clear that the conclusion of the theorem can not be improved significantly if the operator is not elliptic.

(b). In many situations, the microlocal smoothness assumption of the solution can be imposed by appropriate side conditions, such as Cauchy data for hyperbolic problems.

(c). Although in this paper only the *P.D.E.* with smooth coefficients are considered, we claim that it is entirely possible to prove trace theorems for general nonsmooth coefficients cases by analyzing various results on linear propagation of singularities, for example those in Beals and Reed [3] and [4], on this matter partial results have be obtained in Bao [1].

(d). Note that, it is crucial to investigate the trace regularities of solutions to linear Partial differential equations defined only on one side of the trace (corresponding to boundary value problems). The difficulties will present immediately due to the fact that the $\Psi.D.O.$ cutoff technique engaged in the proof of the previous lemma will break down around the trace in general. Nevertheless, one is able to get some partial results along the same line of the proof of the trace theorem as follows.

Corollary. *Assume that $P(x, t, D_x, D_t)$ is a linear partial differential operator of order m with smooth coefficients. Also, assume there is a smooth* *family of* Ψ*.D.O. and a small constant* $\epsilon > 0$,

$$Q(t) = Q(x, t, D_x) \in C^\infty\{[0, \epsilon], OPS^0_{1,0}(\Omega)\} ,$$

with Ω *is an open subset of* $\mathbb{R}^n$*, such that*

(1). *if* $(x, \xi) \in [ES(Q)]^C$*, for all* $0 \le t \le \epsilon$, $P(x, t, \xi, \tau)$ *has* m *distinct real roots as a polynomial of* τ*;*

(2). $(Qu)\mid_{t=t_0} \in H^{s+1}_{loc}(\Omega)$, $\;0 \le t_0 \le \epsilon$,

where $u \in H^s_{loc}(\Omega \times [0, \epsilon])$ *satisfies the equation* $Pu = 0$ *in* $\Omega \times [0, \epsilon]$.

Then, for $\phi(x, t) \in C^\infty_0(\Omega \times [-\epsilon, \epsilon])$,

$$(\phi u)\mid_{t=0} \in H^s(\Omega).$$

Proof of the Corollary. We basically follow the proof of the trace theorem, indicating the necessary modifications. Let us first extend u (in whatever way) to a small neighborhood of the trace such that

$$u \in H^s_{loc}(\Omega \times [-\epsilon, \epsilon]).$$

W.L.O.G. we may assume that $u(x, t)$ has compact support in x. Morever, the pseudolocal property allows one to assume further that Q is compactly supported in x. Therefore, for all $r \in \mathbb{R}$

$$Q : H^r(\Omega) \to H^r(\Omega), \text{ uniformly in } t \in [0, \epsilon].$$

Observe that

$$\tilde{P} = (I - Q)P + QP_0 \stackrel{\text{def}}{=} (I - Q)P + Q \prod_{i=1}^{m} (\tau - \alpha_i \mid \xi \mid)$$

with $\{\alpha_i\}$ are real and distinct constants. It follows that $\tilde{P}$ is strictly hyperbolic $\Psi.D.O.$ of order m and is differential in t. Now we have the following strictly hyperbolic Cauchy problem of ϕu (recall ϕ has compact support),

$$\begin{aligned} \tilde{P}\phi u &= (I - Q)P\phi u + QP_0\phi u \\ &= (I - Q)[P, \phi]u + QP_0\phi u. \end{aligned}$$

Hence, standard hyperbolic energy estimates yield

$$\| \phi u \mid_{t=0} \|^2_{H^s(\Omega)} \leq \int_0^\epsilon \{ \| (I - Q)[P, \phi]u \|^2_{H^{s-(m-1)}(\Omega)} + \| QP_0\phi u \|^2_{H^{s-(m-1)}(\Omega)} \} dt$$

$$\leq C \int_0^\epsilon \{ \| \phi_0 u \|^2_{H^s(\Omega)} + \| [Q, P_0\phi]u \|^2_{H^{s-(m-1)}(\Omega)} + \| P_0\phi Q u \|^2_{H^{s-(m-1)}(\Omega)} \} dt$$

$$\leq C \| \phi_0 u \|^2_{H^s(\Omega \times [-\epsilon, \epsilon])} + C \int_0^\epsilon \| \phi_1 u \|^2_{H^s(\Omega)} \, dt + C \int_0^\epsilon \| P_0 \phi Q u \|^2_{H^{s+1}(\Omega)} \, dt,$$

where ϕ_0 and ϕ_1 are smooth functions with compact support contained in $supp(\phi)$. Notice that, to obtain the above estimates, the continuous property of Q and the fact that the extended u is in $H^{s+1}_{loc}(\Omega \times [-\epsilon, \epsilon])$ have been used. Therefore, in order to accomplish our proof, it suffices to show that

$$\int_0^\epsilon \| \phi Q u \|^2_{H^{s+1}(\Omega)} \, dt < +\infty ,$$

which is exactly our assumption (2). □

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