The Optimal Design of Laminated Beams

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1. Introduction

The use of layered or sandwich structures is now a well–recognized means by which energy may be dissipated in flexible structures. The idea is simple, namely, to alternate layers of stiff elastic material with layers of soft viscous material. As usual however the science of such structures has taken a back seat to the issues of fabrication and specific application. In particular, the mathematical models have been crude and one does not find an application independent characterization of the notion of vibration absorption. The recent work of Hansen and coworkers has dramatically improved the situation *viz a viz* models. Our efforts amount to clarifying the design issues in light of their work. In §2 we present their model of a two layer beam with adhesive damping and our specification of the design problem. In §3 we explain how ARPACK is used to evaluate the objective function and how PDS addresses its minimization. We close in §4 with a short film of the pinned, optimally laminated, beam.

2. The Model and the Design Problem

From Hansen and Spies [1] we extract a model for a two–layer beam in which slip can occur along the interface. It is assumed that an adhesive layer of negligible thickness and mass bonds the two adjoining surfaces in such a way that the restoring force created by the adhesive is proportional to the amount of slip. In particular, the transverse displacement, $y(x,t)$, and the slip, $z(x,t)$, satisfy

$$\begin{aligned} \rho y_{tt} - (Dz_x)_{xx} &= 0, \\ \beta(z_t + y_{tx}) - 3(Dz_x)_x - \frac{3}{4}(Dy_{xx})_x &= 0, \end{aligned} \qquad 0 < x < \ell,\ 0 < t, \tag{1}$$

where ρ, D, and β denote density, flexural rigidity, and adhesive damping. If each layer is a one meter length of steel that is 0.005 meters thick then $\ell = 1$, $\rho = 2000$, and $D = 56250$ are suitable choices. In addition we shall suppose the beam to be pinned at its ends, i.e.,

$$y(0,t) = y_{xx}(0,t) = y(1,t) = y_{xx}(1,t) = 0 \quad \text{and} \quad z_x(0,t) = z_x(1,t) = 0, \quad 0 < t.$$

This leaves the choice of β to the designer. We propose that β be distributed in such a way that the rate of decay of energy is minimized. This decay rate, $\mu(\beta)$, corresponds to the spectral abscissa of the matrix differential operator obtained by writing (1) as a first order system. In particular,

$$\mu(\beta) \equiv \sup_{\lambda \in \sigma(\beta)} \Re\lambda, \tag{2}$$

where $\sigma(\beta)$ is the spectrum of

$$A(\beta) = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & \frac{D}{\rho} d_x^3 \\ \frac{3D}{4\beta} d_x^2 & -d_x & \frac{3D}{\beta} d_x^2 \end{pmatrix}. \tag{3}$$

If β is a positive function then the spectrum of $A(\beta)$ will lie in the left halfplane and so $\mu(\beta)$ will be negative. The design objective is to make it as negative as possible. More precisely, we address

$$\inf_{\beta \in L^\infty(0,1)} \mu(\beta). \tag{4}$$

Though $\mu(\beta)$ may appear as an obvious choice of objective function one rarely finds it in the literature, perhaps for the reason that it suffers doubly, being neither Lipschitz, nor easy to estimate. We address these obstacles in the next section.

3. The Algorithm

We estimate $\mu(\beta)$ by applying ARPACK to a Galerkin–spectral discretization of $A(\beta)$. We chose the spectral method over both finite differences and elements for the reason that the improvement in accuracy outweighed the cost of working with dense linear systems. The Galerkin basis of choice was

$$\{V_j, \overline{V}_j\}_{j=1}^m, \quad V_j = \frac{1}{\sqrt{2} j^2 \pi^2} \begin{pmatrix} \sin(j\pi x) \\ i j^2 \pi^2 \sin(j\pi x) \\ j\pi\sqrt{2}\cos(j\pi x) \end{pmatrix}. \tag{5}$$

We denote by β_n a piecewise constant approximation to β at n points and denote by $A_m(\beta_n)$ the nonnormal, complex, dense, matrix of order $2m$ that results from the Galerkin discretization of $A(\beta)$. Rather than computing the entire spectrum of $A_m(\beta_n)$ only to keep the element with the largest real part we invoke, with care, the shift and invert strategy offered by ARPACK and effectively compute the half dozen or so right–most eigenvalues. For m on the order of 64 this method was faster, by a factor of 10, than the LAPACK approach.

With regard to the minimization of $\mu(A_m(\beta_n))$ we recall the excellent performance of PDS on the related though simpler problems (shear buildings and strings) treated in [2]. The fact that $\mu(A_m(\beta_n))$ does not possess a bounded derivative in a neighborhood of its minimizer all but forces us to exploit derivative–free search techniques.

4. Numerical Results

With 64 basis functions, $m = 32$, and a partition of β into 32 segments, $n = 32$, PDS converged from a random starting point in 52 iterations to a tolerance of $1e-6$. Thirtynine minutes (wall clock) elapsed during its 68935 function evaluations. It produced an optimal β that is close to a constant value of 11195. The two figures on the following page depict the associated spectrum. The movie depicts the optimally damped beams rapid return to rest after being subjected to an initial disturbance.

5. References

[1] Hansen, S.W. and Spies, R.D., *Structural Damping in Laminated Beams Due to Interfacial Slip*, J. Sound & Vibration, submitted.

[2] Cox, S., *Designing for Optimal Energy Absorption, III: Numerical Minimization of the Spectral Abscissa*, International J. Structural & Multidisciplinary Optimization, submitted.