Investigating the Effective Return Rate for Two Investment Strategies: Lump Sum and Dollar-Cost Averaging

Pedro G. Sepulveda, Jr.
Department of Computer and Mathematical Sciences
University of Houston-Downtown

***Abstract*— This paper compares two investment strategies, Dollar-Cost Averaging and Lump Sum Investing, via a log-normal asset model simulated with varying annual percentage returns, standard deviation, and investment duration. It describes the methodology of both strategies along with specific examples. The results of the paper describe the counterintuitive conclusion that was reached via the project's computational simulation.**

I. Introduction

The goal of this project was to compare the effectiveness of two distinct investment strategies: Dollar-Cost Averaging and Lump Sum Investing. Both financial strategies have a specific term: a finite time period from the start of the investment to the end. In this project the intervals of each investment were measured monthly. Both strategies, however, have a different investment methodology. Dollar-Cost Averaging (DCA) refers to making an equal investment each period of the term. Lump Sum Investing (LSI) refers to making one initial investment and waiting until the end of the term. Because of the difference in approach, there cannot be a direct comparison just between the profits of each method. Thus, the corresponding return rates have to be computed in order to compare both methods. The comparison computation was effectively done using a computer and mathematical simulation approach.

II. Model

In order to compare both investment strategies, a similar model had to be used to simulate the methods. The comparison between the Lump Sum Investment and Dollar-Cost Averaging was simulated using the following model for the asset return r:

$$1 + r \; = LN(\mu, \sigma^2)$$

thus $1+r$ is independent and identically distributed, with a log-normal distribution. The assumption of this model was that there are no correlations nor trends between each time interval of an investment. The simulation ran 120 cases with 10,000 random return rates per case using a combination of the date on Table 1.

Variable	Data	Description
t	1,2,4,8,16,32	number of years
μ	3,6,9,12,15	annual return in %
σ	$\mu/2, \mu, 2\mu, 4\mu$	standard deviation

Table 1: Input Data

III. Methods

A. Lump Sum Investment

In the Lump Sum Investment case, the two factors that are necessary to calculate the final balance of an investment are the initial investment amount and the number of years of an investment . Given the description of the LSI case, the following algorithm was constructed using log-normal input values for the returns r_n :

$$A = P(1 + r_1)(1 + r_2)(1 + r_3)...(1 + r_N)$$

This algorithm is mathematically simplified into the following

$$A = P \prod_{n=1}^{N} (1 + r_n)$$

where r_n is a random rate of return, $N = 12 * t$ is the number of monthly periods, P is the initial investment, and A is the final balance. An example of the LSI method would be as follows: given a three month investment ($N = 3$) where the initial deposit was $\$1,000.00$ ($P = 1000$), and rates of return where $r_1 = 10\%, r_2 = 20\%, r_3 = 5\%$, then the final amount would be calculated as :

$$A = 1000(1 + 0.10)(1 + 0.20)(1 + 0.05) = 1386$$

The effective rate $(1 + r)$ is the one r that would give the same final balance if all the r_n were equal.

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For the LSI investment strategy, the effective return rate r can be calculated using the following formula:

$$r = (A/P)^{1/n} - 1$$

The effective return rate of the example above is 11.5% per month, which is positive.

If we assume that all values of $R_n = i$, meaning the return rates are fixed for all n values, then

$$A = P(1+i)^n.$$

This formula is a special case with the LSI investment methodology. A common example of this would be investing capital in a savings account which has a fixed interest rate. For example, given a three month investment ($n = 3$) where the initial deposit was \$1000.00 ($P = 1000$) as in the first example, but with a constant interest rate r_n of 10% ($i = 0.10$), the final amount would be calculated as:

$$A = 1000(1+0.10)^3 = 1331$$

Notice that the effective rate $r = (A/P)^{1/n} - 1$ is positive for this example, in other words, $r = 10\%$.

B. Dollar-Cost Averaging

The input for the rate of return values r_n are the same as in the Lump Sum Investment case: asset return rates are still $LN(\mu, \sigma^2)$. The fundamental difference between a Dollar-Cost Average investment and the LSI approach is the addition of the periodic or monthly investment. Thus, the algorithm for the DCA case changes slightly compared to the LSI case. Lets assume the following general investment example: P is the initial balance. After the first month, the balance is

$$A = P(1+r_1) + P$$

After the second month, the balance is

$$A = P(1+r_1)(1+r_2) + P(1+r_2) + P$$

Then, after the Nth month, the final balance would be

$$A = P\prod_{n=1}^{N}(1+r_n) + \prod_{n=2}^{N}(1+r_n) + \ldots + \prod_{n=N}^{N}(1+r_n) + P$$

Thus, this algorithm is mathematically simplified into the following

$$A = P\sum_{k=1}^{N+1}\prod_{n=k}^{N}(1+r_n)$$

The effective rate $(1+r)$ is still the one r that would give the same final balance if all the r_n were equal. With DCA the effective return rate is not easily computed because of the sums of the products of the return rates in the algorithm. Thus, in order to approximate the effective return rate $(1+r)$, Newton's Iterative Method is used with a given initial guess near the input μ and the following mathematical relation

$$A = P((1+r)^{N+1} - 1)/r$$

As an example of the Dollar-Cost Average approach, let us look at the following example. Similar to the examples shown before, an investment is made for three months ($N = 3$) where the initial deposit as well as the monthly investment is \$1000.00 ($P = 1000$) and rates of return are $r_1 = 10\%, r_2 = 20\%, r_3 = 5\%$, then the final amount would be calculated as:

$$A = 1000(1+0.10)(1+0.20)(1+0.05) + 1000(1+0.20)(1+0.05) + 1000(1+0.05) + 1000 = 4696$$

Using Newton's Method, the effective return rate for this DCA example is approximately 10.8%, which in this instance is lower than the LSI example, even though the final balance was higher.

IV. Results

The goal of this project was to compare Dollar-Cost Average and Lump Sum Investment by analyzing the effective return rates of each strategy using computational mathematics. In the financial press, it is a common belief that Dollar-Cost Average is less of a risk then the Lump Sum Investment approach. The data that resulted from this project contradict this general investment belief.

The first interesting comparison was made between the LSI standard deviation (LSI stdev) and the computed log-normal sigma value (LN sigma). A graphical illustration of this comparison is shown in Figure 1 on page 6.

To verify the illustration, a mathematical relation was formulated to determine the expectation of the data. Using the standard deviation relation

$$StandardDeviation = m\sqrt{w(w-1)}$$

where $m = e^{\mu}$ and $w = e^{\sigma^2/N}$ the expectation comparison between the LSI standard deviation and the log-normal sigma would approximately be

$$LSIstdev \ = \sigma/\sqrt{N}$$

For example, if a one year investment ($N = 1$) was made with a log-normal sigma value of 0.024 ($\sigma = 0.024$), then the LSI standard deviation would approximately be 0.0069 which does reflect the relationship shown on Figure 1.

The most significant comparison that resulted from this project was the comparison between the Dollar-Cost Average and the Lump Sum Investment strategies. The comparison was made using both the calculated μ and σ of each investment strategy. In Figure 2 on page 7, the base of each arrow represents the Lump Sum Investment point (μ, σ) and the tip of the arrow represents the Dollar-Cost Average point (μ, σ). The main interpretation of the computed data was that the Dollar-Cost Average was more of a risk for not much more of a return compared to the Lump Sum Investment approach.

The project hypothesis was that Dollar-Cost Averaging was less risky than the Lump Sum approach. The mean of the DCA approach was expected to be either equal to or greater than the LSI mean with definitely less of a standard deviation. Although it was expected that Dollar-Cost Averaging would be the safer approach, the computational simulation of these two strategies yielded the contrary conclusion.

V. Concluding Remarks

Investing money using Lump Sum Investment or any other strategy of course depends on an individual's financial capacity. But the utilization of mathematics along with computational analysis resulted in a counterintuitive conclusion about the main hypothesis of this project. Mathematical computation using a simulation approach is extremely essential in order to understand aspects of the real world that are not concretely known or understood. Even non-mathematical intuition as in the case with this project is subject to mathematical interpretation. The results of any real-world simulation may yield a better understanding of the problem being simulated, even if it results in a counterintuitive conclusion, as occured in this particular project.