Optimizing Cobb-Douglas Production Function in Economics

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***Abstract*— The Cobb-Douglas production function is a simple mathematical model in economics. The function provides the relationship among the number of units produced by different amounts of labor and capital.**

Two applied optimization problems in economics with Cobb-Douglas model are studied in this report. Solutions are provided as well.

I. Cobb-Douglas Production Function

A. Production function

In economics, "production" means that any process which converts or transforms a commodity or commodities into a different commodity. A general statement of all outputs that can be obtained from all efficient input combinations is called the *production function.*

A typical production function may involve two inputs *Labor* and *Capital.* If we use P, L, and C to represent units of Production, units of Labor and units of Capital respectively, then the production function in general can be written as $P = F(L, C)$. However, production functions are required to have certain properties in economics [1].

B. Cobb-Douglas Production Function:

To model the nations economy, Douglas fitted data with the log-linear function $\log P = \log A + \alpha \log L + \beta \log C$ suggested by Cobb, where A, α, and β are constant parameters. The log-linear function is mathematically equivalent to

$$P = AL^{\alpha}C^{\beta} \tag{1}$$

Equation (1) is the well known Cobb-Douglas production function in economics. It is the simplest mathematical model with *all essential properties of production functions.* From economics [1], we have

$$\alpha + \beta = 1 \tag{2}$$

If we use R, a and b to represent total cost, per unit cost of labor, and per unit cost of capital, respectively, then

$$R = aL + bC \tag{3}$$

We study optimization problems related to equations (1) and (3).

II. Optimization Problems of Cobb-Douglas Function:

A. A Maximization Problem

In practice, a manufacturer has only limited resources for the total cost of labor and capital. If each unit of products makes the same amount of profit, then the manufacturer would like to reasonably locate available resources to maximize the number of products it produces.

This problem can be expressed in mathematical form as

Maximize: $P = AL^{\alpha}C^{\beta}$
Subject to: $R = aL + bC$

where R, a and b are given, and represent total available resources, per unit cost of labor, and per unit cost of capital, respectively.

B. A Minimization Problem:

It may not be true that each unit of products makes the same amount of profit. Through market analysis on demanding and supply, a manufacture probably should only produce certain amount of products to maximize its profit. In this case, we want to minimize the total cost on labor and capital.

In mathematical form, the problem can be stated as:

Minimize: $R = aL + bC$
Subject to: $P = AL^{\alpha}C^{\beta}$.

In the above, P, a and b are given, and represent total units of production, per unit cost of labor, and per unit cost of capital, respectively.

III. Mathematical Tools

Both of the above two optimization problems require us to find extrema for a function $z = f(x, y)$ subject to a constraint $g(x, y) = c$. Such kind of optimization problems had been studied by Joseph L.

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Lagrange, an Italian mathematician in the 18-19th century. Lagrange published his result, now known as the Lagrange theorem [2], when he was 19 year old as a professor at the Royal Artillery School in Turin, Italy.

Lagrange Theorem: If both $f(x, y)$ and $g(x, y)$ have continuous first partial derivatives such that $f(x, y)$ has an extremum at a point (x_0, y_0) on the constraint $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq 0$, then there is a real number λ, now called Lagrange multiplier, such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Based on the theorem, an algorithm called the Method of Lagrange's Multiplier [2] has been developed to solve constraint optimization problems. We use this method to solve the two optimization problems in economics with Cobb-Douglas production function.

IV. Solutions

A. Solution for the Maximization Problem:

By Lagrange theorem, we have $\frac{\partial P}{\partial L} = \lambda \frac{\partial R}{\partial L}$, and $\frac{\partial P}{\partial C} = \lambda \frac{\partial R}{\partial C}$ for equations (1) and (3) at the maximum. Therefore,

$$\alpha A L^{\alpha-1} C^{\beta} = \lambda a \tag{4}$$

and

$$\beta A L^{\alpha} C^{\beta-1} = \lambda b \tag{5}$$

Dividing equation (4) by (5), we obtain

$$\alpha b C = \beta a L \tag{6}$$

Combining (6) with the constraint $R = aL + bC$, and using (2), we obtain
$L = \frac{\alpha R}{a}$ and $C = \frac{\beta R}{b}$.

We can also obtain the Lagrange multiplier $\lambda = \frac{\alpha^{\alpha} \beta^{\beta}}{a^{\alpha} b^{\beta}} A$.

Since A, R, a, b, α, and β are given, the manufacturer can determine the L and C to maximize the production with available resources R. The maximum production will be $P = \frac{\alpha^{\alpha} \beta^{\beta}}{a^{\alpha} b^{\beta}} AR = \lambda R$. The Lagrange multiplier λ represents that at the optimal case the product increasing by per dollar investment. In economics, it is also called the *marginal productivity of money*.

B. Solution for the Minimization Problem:

Similarly, we can solve the minimization problem by using (3) as the objective function and (1) as the constraint. From Lagrange theorem, we have $\frac{\partial R}{\partial L} = \lambda \frac{\partial P}{\partial L}$, and $\frac{\partial R}{\partial C} = \lambda \frac{\partial P}{\partial C}$ at the minimum. Therefore,

$$a = \lambda \alpha A L^{\alpha-1} C^{\beta} \tag{7}$$

$$b = \lambda \beta A L^{\alpha} C^{\beta-1} \tag{8}$$

In the three equations (7), (8), and the constraint $P = A L^{\alpha} C^{\beta}$, A, P, a, b, α, and β are known. By solving for λ, L, and C, we obtain $\lambda = \frac{a^{\alpha} b^{\beta}}{A \alpha^{\alpha} \beta^{\beta}}$, $L = \frac{P}{A} (\frac{\alpha b}{\beta a})^{\beta}$ and $C = \frac{P}{A} (\frac{\beta a}{\alpha b})^{\alpha}$.

Now, the manufacturer can use the L and C to make P production with minimized total cost. The minimized cost will be $R = \frac{P a^{\alpha} b^{\beta}}{A \alpha^{\alpha} \beta^{\beta}} = \lambda P$. The Lagrange multiplier λ represents the cost per unit production at the optimal case. It is called the *marginal cost of productivity.*

V. Conclusions and Questions:

By completing this project, we obtained solutions for two applied optimization problems in economics with the method of Lagrange multiplier. Mathematics provides us powerful tools to solve real word problems in science and engineering.

In this project, we used Cobb-Douglas production function as the mathematical model. However, I still have some questions such as how precisely Cobb-Douglas function models the economy? how can we find the parameters A, α, and β in Cobb-Douglas function for a specific manufacturer? Further study is needed to answer these questions for me.

References

[1] Kelvin Lancaster, Introduction to Modern Micro Economics, Rand McNally & Company.

[2] James Stewart, Calculus, Brooks/Cole Publishing Company.